10

Coupled Lines

10.1 Coupled Transmission Lines

Coupling between two transmission lines is introduced by their proximity to each other. Coupling effects may be undesirable, such as *crosstalk* in printed circuits, or they may be desirable, as in *directional couplers* where the objective is to transfer power from one line to the other.

In Sections 10.1–10.3, we discuss the equations, and their solutions, describing coupled lines and crosstalk [486–503]. In Sec. 10.4, we discuss directional couplers, as well as fiber Bragg gratings, based on coupled-mode theory [504–525]. Fig. 10.1.1 shows an example of two coupled microstrip lines over a common ground plane, and also shows a generic circuit model for coupled lines.

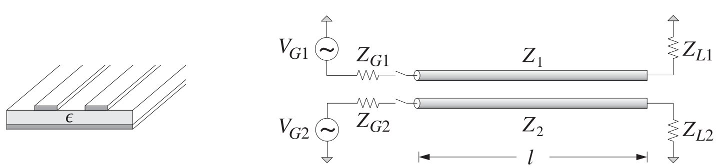


Fig. 10.1.1 Coupled Transmission Lines.

For simplicity, we assume that the lines are lossless. Let L_i, C_i, $i = 1, 2$ be the distributed inductances and capacitances *per unit length* when the lines are isolated from each other. The corresponding propagation velocities and characteristic impedances are: $v_i = 1/\sqrt{L_i C_i}$, $Z_i = \sqrt{L_i/C_i}$, $i = 1, 2$. The coupling between the lines is modeled by introducing a *mutual* inductance and capacitance per unit length, L_m, C_m. Then, the coupled versions of telegrapher's equations (9.15.1) become:†

† C_1 is related to the capacitance to ground C_{1g} via $C_1 = C_{1g} + C_m$, so that the total charge per unit length on line-1 is $Q_1 = C_1 V_1 - C_m V_2 = C_{1g}(V_1 - V_g) + C_m(V_1 - V_2)$, where $V_g = 0$.

$$\begin{aligned} \frac{\partial V_1}{\partial z} &= -L_1\frac{\partial I_1}{\partial t} - L_m\frac{\partial I_2}{\partial t}, & \frac{\partial I_1}{\partial z} &= -C_1\frac{\partial V_1}{\partial t} + C_m\frac{\partial V_2}{\partial t} \\ \frac{\partial V_2}{\partial z} &= -L_2\frac{\partial I_2}{\partial t} - L_m\frac{\partial I_1}{\partial t}, & \frac{\partial I_2}{\partial z} &= -C_2\frac{\partial V_2}{\partial t} + C_m\frac{\partial V_1}{\partial t} \end{aligned} \tag{10.1.1}$$

When $L_m = C_m = 0$, they reduce to the uncoupled equations describing the isolated individual lines. Eqs. (10.1.1) may be written in the 2×2 matrix forms:

$$\begin{aligned} \frac{\partial \boldsymbol{V}}{\partial z} &= -\begin{bmatrix} L_1 & L_m \\ L_m & L_2 \end{bmatrix}\frac{\partial \boldsymbol{I}}{\partial t} \\ \frac{\partial \boldsymbol{I}}{\partial z} &= -\begin{bmatrix} C_1 & -C_m \\ -C_m & C_2 \end{bmatrix}\frac{\partial \boldsymbol{V}}{\partial t} \end{aligned} \tag{10.1.2}$$

where $\boldsymbol{V}, \boldsymbol{I}$ are the column vectors:

$$\boldsymbol{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \quad \boldsymbol{I} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{10.1.3}$$

For sinusoidal time dependence $e^{j\omega t}$, the system (10.1.2) becomes:

$$\begin{aligned} \frac{d\boldsymbol{V}}{dz} &= -j\omega\begin{bmatrix} L_1 & L_m \\ L_m & L_2 \end{bmatrix}\boldsymbol{I} \\ \frac{d\boldsymbol{I}}{dz} &= -j\omega\begin{bmatrix} C_1 & -C_m \\ -C_m & C_2 \end{bmatrix}\boldsymbol{V} \end{aligned} \tag{10.1.4}$$

It proves convenient to recast these equations in terms of the forward and backward waves that are normalized with respect to the uncoupled impedances Z_1, Z_2:

$$\begin{aligned} a_1 &= \frac{V_1 + Z_1 I_1}{\sqrt{2Z_1}}, & b_1 &= \frac{V_1 - Z_1 I_1}{\sqrt{2Z_1}} \\ a_2 &= \frac{V_2 + Z_2 I_2}{\sqrt{2Z_2}}, & b_2 &= \frac{V_2 - Z_2 I_2}{\sqrt{2Z_2}} \end{aligned} \quad\Rightarrow\quad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{10.1.5}$$

The $\mathbf{a}, \mathbf{b}$ waves are similar to the power waves defined in Sec. 12.7. The total average power on the line can be expressed conveniently in terms of these:

$$\begin{aligned} P &= \frac{1}{2}\mathrm{Re}[\boldsymbol{V}^\dagger \boldsymbol{I}] = \frac{1}{2}\mathrm{Re}[V_1^* I_1] + \frac{1}{2}\mathrm{Re}[V_2^* I_2] = P_1 + P_2 \\ &= \big(|a_1|^2 - |b_1|^2\big) + \big(|a_2|^2 - |b_2|^2\big) = \big(|a_1|^2 + |a_2|^2\big) - \big(|b_1|^2 + |b_2|^2\big) \\ &= \mathbf{a}^\dagger\mathbf{a} - \mathbf{b}^\dagger\mathbf{b} \end{aligned} \tag{10.1.6}$$

where the dagger operator denotes the conjugate-transpose, for example, $\mathbf{a}^\dagger = [a_1^*, a_2^*]$. Thus, the $\mathbf{a}$-waves carry power forward, and the $\mathbf{b}$-waves, backward. After some algebra, it can be shown that Eqs. (10.1.4) are equivalent to the system:

$$\boxed{\begin{aligned} \frac{d\mathbf{a}}{dz} &= -jF\,\mathbf{a} + jG\,\mathbf{b} \\ \frac{d\mathbf{b}}{dz} &= -jG\,\mathbf{a} + jF\,\mathbf{b} \end{aligned}} \quad\Rightarrow\quad \frac{d}{dz}\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = -j\begin{bmatrix} F & -G \\ G & -F \end{bmatrix}\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \tag{10.1.7}$$

with the matrices F, G given by:

$$F = \begin{bmatrix} \beta_1 & \kappa \\ \kappa & \beta_2 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & \chi \\ \chi & 0 \end{bmatrix} \tag{10.1.8}$$

where β_1, β_2 are the uncoupled wavenumbers $\beta_i = \omega/v_i = \omega\sqrt{L_i C_i}$, $i = 1, 2$ and the coupling parameters κ, χ are:

$$\begin{aligned} \kappa &= \frac{1}{2}\omega\left(\frac{L_m}{\sqrt{Z_1 Z_2}} - C_m\sqrt{Z_1 Z_2}\right) = \frac{1}{2}\sqrt{\beta_1\beta_2}\left(\frac{L_m}{\sqrt{L_1 L_2}} - \frac{C_m}{\sqrt{C_1 C_2}}\right) \\ \chi &= \frac{1}{2}\omega\left(\frac{L_m}{\sqrt{Z_1 Z_2}} + C_m\sqrt{Z_1 Z_2}\right) = \frac{1}{2}\sqrt{\beta_1\beta_2}\left(\frac{L_m}{\sqrt{L_1 L_2}} + \frac{C_m}{\sqrt{C_1 C_2}}\right) \end{aligned} \tag{10.1.9}$$

A consequence of the structure of the matrices F, G is that the total power P defined in (10.1.6) is conserved along z. This follows by writing the power in the following form, where I is the 2×2 identity matrix:

$$P = \mathbf{a}^\dagger\mathbf{a} - \mathbf{b}^\dagger\mathbf{b} = [\mathbf{a}^\dagger, \mathbf{b}^\dagger]\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}$$

Using (10.1.7), we find:

$$\frac{dP}{dz} = j[\mathbf{a}^\dagger, \mathbf{b}^\dagger]\left(\begin{bmatrix} F^\dagger & G^\dagger \\ -G^\dagger & -F^\dagger \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} F & -G \\ G & -F \end{bmatrix}\right)\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = 0$$

the latter following from the conditions $F^\dagger = F$ and $G^\dagger = G$. Eqs. (10.1.6) and (10.1.7) form the basis of coupled-mode theory.

Next, we specialize to the case of two *identical* lines that have $L_1 = L_2 \equiv L_0$ and $C_1 = C_2 \equiv C_0$, so that $\beta_1 = \beta_2 = \omega\sqrt{L_0 C_0} \equiv \beta$ and $Z_1 = Z_2 = \sqrt{L_0/C_0} \equiv Z_0$, and speed $v_0 = 1/\sqrt{L_0 C_0}$. Then, the $\mathbf{a}, \mathbf{b}$ waves and the matrices F, G take the simpler forms:

$$\mathbf{a} = \frac{\boldsymbol{V} + Z_0\boldsymbol{I}}{\sqrt{2Z_0}}, \quad \mathbf{b} = \frac{\boldsymbol{V} - Z_0\boldsymbol{I}}{\sqrt{2Z_0}} \quad \Rightarrow \quad \mathbf{a} = \frac{\boldsymbol{V} + Z_0\boldsymbol{I}}{2}, \quad \mathbf{b} = \frac{\boldsymbol{V} - Z_0\boldsymbol{I}}{2} \tag{10.1.10}$$

$$F = \begin{bmatrix} \beta & \kappa \\ \kappa & \beta \end{bmatrix}, \quad G = \begin{bmatrix} 0 & \chi \\ \chi & 0 \end{bmatrix} \tag{10.1.11}$$

where, for simplicity, we removed the common scale factor $\sqrt{2Z_0}$ from the denominator of $\mathbf{a}, \mathbf{b}$. The parameters κ, χ are obtained by setting $Z_1 = Z_2 = Z_0$ in (10.1.9):

$$\kappa = \frac{1}{2}\beta\left(\frac{L_m}{L_0} - \frac{C_m}{C_0}\right), \quad \chi = \frac{1}{2}\beta\left(\frac{L_m}{L_0} + \frac{C_m}{C_0}\right), \tag{10.1.12}$$

The matrices F, G commute with each other. In fact, they are both examples of matrices of the form:

$$A = \begin{bmatrix} a_0 & a_1 \\ a_1 & a_0 \end{bmatrix} = a_0 I + a_1 J, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{10.1.13}$$

where a_0, a_1 are real such that $|a_0| \neq |a_1|$. Such matrices form a *commutative* subgroup of the group of nonsingular 2×2 matrices. Their eigenvalues are $\lambda_\pm = a_0 \pm a_1$ and they can all be diagonalized by a *common* unitary matrix:

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = [\mathbf{e}_+, \mathbf{e}_-], \quad \mathbf{e}_+ = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{e}_- = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{10.1.14}$$

so that we have $QQ^\dagger = Q^\dagger Q = I$ and $A\mathbf{e}_\pm = \lambda_\pm\mathbf{e}_\pm$.

The eigenvectors $\mathbf{e}_\pm$ are referred to as the *even* and *odd* modes. To simplify subsequent expressions, we will denote the eigenvalues of A by $A_\pm = a_0 \pm a_1$ and the diagonalized matrix by $\bar{A}$. Thus,

$$A = Q\bar{A}Q^\dagger, \quad \bar{A} = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix} = \begin{bmatrix} a_0 + a_1 & 0 \\ 0 & a_0 - a_1 \end{bmatrix} \tag{10.1.15}$$

Such matrices, as well as any matrix-valued function thereof, may be diagonalized simultaneously. Three examples of such functions appear in the solution of Eqs. (10.1.7):

$$\begin{aligned} \mathcal{B} &= \sqrt{(F+G)(F-G)} = Q\sqrt{(\bar{F}+\bar{G})(\bar{F}-\bar{G})}\,Q^\dagger \\ Z &= Z_0\sqrt{(F+G)(F-G)^{-1}} = Z_0 Q\sqrt{(\bar{F}+\bar{G})(\bar{F}-\bar{G})^{-1}}\,Q^\dagger \\ \Gamma &= (Z - Z_0 I)(Z + Z_0 I)^{-1} = Q(\bar{Z} - Z_0 I)(\bar{Z} + Z_0 I)^{-1}Q^\dagger \end{aligned} \tag{10.1.16}$$

Using the property $FG = GF$, and differentiating (10.1.7) one more time, we obtain the decoupled second-order equations, with $\mathcal{B}$ as defined in (10.1.16):

$$\frac{d^2\mathbf{a}}{dz^2} = -\mathcal{B}^2\,\mathbf{a}, \quad \frac{d^2\mathbf{b}}{dz^2} = -\mathcal{B}^2\,\mathbf{b}$$

However, it is better to work with (10.1.7) directly. This system can be decoupled by forming the following linear combinations of the $\mathbf{a}, \mathbf{b}$ waves:

$$\begin{aligned} \boldsymbol{A} &= \mathbf{a} - \Gamma\mathbf{b} \\ \boldsymbol{B} &= \mathbf{b} - \Gamma\mathbf{a} \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{bmatrix} = \begin{bmatrix} I & -\Gamma \\ -\Gamma & I \end{bmatrix}\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \tag{10.1.17}$$

The $\boldsymbol{A}, \boldsymbol{B}$ can be written in terms of $\boldsymbol{V}, \boldsymbol{I}$ and the impedance matrix Z as follows:

$$\begin{aligned} \boldsymbol{A} &= (2D)^{-1}(\boldsymbol{V} + Z\boldsymbol{I}) \\ \boldsymbol{B} &= (2D)^{-1}(\boldsymbol{V} - Z\boldsymbol{I}) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \boldsymbol{V} &= D(\boldsymbol{A} + \boldsymbol{B}) \\ Z\boldsymbol{I} &= D(\boldsymbol{A} - \boldsymbol{B}) \end{aligned} \qquad D = \frac{Z + Z_0 I}{2Z_0} \tag{10.1.18}$$

Using (10.1.17), we find that $\boldsymbol{A}, \boldsymbol{B}$ satisfy the decoupled first-order system:

$$\frac{d}{dz}\begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{bmatrix} = -j\begin{bmatrix} \mathcal{B} & 0 \\ 0 & -\mathcal{B} \end{bmatrix}\begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{bmatrix} \quad \Rightarrow \quad \frac{d\boldsymbol{A}}{dz} = -j\mathcal{B}\boldsymbol{A}, \quad \frac{d\boldsymbol{B}}{dz} = j\mathcal{B}\boldsymbol{B} \tag{10.1.19}$$

with solutions expressed in terms of the matrix exponentials $e^{\pm j\mathcal{B}z}$:

$$\boldsymbol{A}(z) = e^{-j\mathcal{B}z}\boldsymbol{A}(0), \quad \boldsymbol{B}(z) = e^{j\mathcal{B}z}\boldsymbol{B}(0) \tag{10.1.20}$$

Using (10.1.18), we obtain the solutions for $\boldsymbol{V}, \boldsymbol{I}$:

$$\begin{aligned} \boldsymbol{V}(z) &= D\big[e^{-j\mathcal{B}z}\boldsymbol{A}(0)+e^{j\mathcal{B}z}\boldsymbol{B}(0)\big] \\ Z\boldsymbol{I}(z) &= D\big[e^{-j\mathcal{B}z}\boldsymbol{A}(0)-e^{j\mathcal{B}z}\boldsymbol{B}(0)\big] \end{aligned} \tag{10.1.21}$$

To complete the solution, we assume that both lines are terminated at common generator and load impedances, that is, $Z_{G1} = Z_{G2} \equiv Z_G$ and $Z_{L1} = Z_{L2} \equiv Z_L$. The generator voltages V_{G1}, V_{G2} are assumed to be different. We define the generator voltage vector and source and load matrix reflection coefficients:

$$\boldsymbol{V}_G = \begin{bmatrix} V_{G1} \\ V_{G2} \end{bmatrix}, \qquad \begin{aligned} \Gamma_G &= (Z_G I - Z)(Z_G I + Z)^{-1} \\ \Gamma_L &= (Z_L I - Z)(Z_L I + Z)^{-1} \end{aligned} \tag{10.1.22}$$

The terminal conditions for the line are at $z = 0$ and $z = l$:

$$\boldsymbol{V}_G = \boldsymbol{V}(0)+Z_G\boldsymbol{I}(0)\,, \quad \boldsymbol{V}(l) = Z_L\boldsymbol{I}(l) \tag{10.1.23}$$

They may be re-expressed in terms of $\boldsymbol{A}, \boldsymbol{B}$ with the help of (10.1.18):

$$\boldsymbol{A}(0)-\Gamma_G\boldsymbol{B}(0) = D^{-1}Z(Z+Z_G I)^{-1}\boldsymbol{V}_G\,, \quad \boldsymbol{B}(l) = \Gamma_L\boldsymbol{A}(l) \tag{10.1.24}$$

But from (10.1.19), we have:†

$$e^{j\mathcal{B}l}\boldsymbol{B}(0) = \boldsymbol{B}(l) = \Gamma_L\boldsymbol{A}(l) = \Gamma_L e^{-j\mathcal{B}l}\boldsymbol{A}(0) \quad\Rightarrow\quad \boldsymbol{B}(0) = \Gamma_L e^{-2j\mathcal{B}l}\boldsymbol{A}(0) \tag{10.1.25}$$

Inserting this into (10.1.24), we may solve for $\boldsymbol{A}(0)$ in terms of the generator voltage:

$$\boldsymbol{A}(0) = D^{-1}\big[I-\Gamma_G\Gamma_L e^{-2j\mathcal{B}l}\big]^{-1}Z(Z+Z_G I)^{-1}\boldsymbol{V}_G \tag{10.1.26}$$

Using (10.1.26) into (10.1.21), we finally obtain the voltage and current at an arbitrary position z along the lines:

$$\boxed{\begin{aligned} \boldsymbol{V}(z) &= \big[e^{-j\mathcal{B}z}+\Gamma_L e^{-2j\mathcal{B}l}e^{j\mathcal{B}z}\big]\big[I-\Gamma_G\Gamma_L e^{-2j\mathcal{B}l}\big]^{-1}Z(Z+Z_G I)^{-1}\boldsymbol{V}_G \\ \boldsymbol{I}(z) &= \big[e^{-j\mathcal{B}z}-\Gamma_L e^{-2j\mathcal{B}l}e^{j\mathcal{B}z}\big]\big[I-\Gamma_G\Gamma_L e^{-2j\mathcal{B}l}\big]^{-1}(Z+Z_G I)^{-1}\boldsymbol{V}_G \end{aligned}} \tag{10.1.27}$$

These are the coupled-line generalizations of Eqs. (9.9.7). Resolving $\boldsymbol{V}_G$ and $\boldsymbol{V}(z)$ into their even and odd modes, that is, expressing them as linear combinations of the eigenvectors $\mathbf{e}_\pm$, we have:

$$\begin{aligned} \boldsymbol{V}_G &= V_{G+}\mathbf{e}_+ + V_{G-}\mathbf{e}_-\,, \quad \text{where} \quad V_{G\pm} = \frac{V_{G1}\pm V_{G2}}{\sqrt{2}} \\ \boldsymbol{V}(z) &= V_+(z)\mathbf{e}_+ + V_-(z)\mathbf{e}_-\,, \qquad V_\pm(z) = \frac{V_1(z)\pm V_2(z)}{\sqrt{2}} \end{aligned} \tag{10.1.28}$$

In this basis, the matrices in (10.1.27) are diagonal resulting in the equivalent solution:

$$\begin{aligned} \boldsymbol{V}(z) = V_+(z)\mathbf{e}_+ + V_-(z)\mathbf{e}_- &= \frac{e^{-j\beta_+ z}+\Gamma_{L+}e^{-2j\beta_+ l}e^{j\beta_+ z}}{1-\Gamma_{G+}\Gamma_{L+}e^{-2j\beta_+ l}}\,\frac{Z_+}{Z_++Z_G}\,V_{G+}\mathbf{e}_+ \\ &+ \frac{e^{-j\beta_- z}+\Gamma_{L-}e^{-2j\beta_- l}e^{j\beta_- z}}{1-\Gamma_{G-}\Gamma_{L-}e^{-2j\beta_- l}}\,\frac{Z_-}{Z_-+Z_G}\,V_{G-}\mathbf{e}_- \end{aligned} \tag{10.1.29}$$

†The matrices $D, Z, \Gamma_G, \Gamma_L, \Gamma, \mathcal{B}$ all commute with each other.

where $\beta_\pm$ are the eigenvalues of $\mathcal{B}$, $Z_\pm$ the eigenvalues of Z, and $\Gamma_{G\pm}, \Gamma_{L\pm}$ are:

$$\Gamma_{G\pm} = \frac{Z_G - Z_\pm}{Z_G + Z_\pm}\,, \quad \Gamma_{L\pm} = \frac{Z_L - Z_\pm}{Z_L + Z_\pm} \tag{10.1.30}$$

The voltages $V_1(z), V_2(z)$ are obtained by extracting the top and bottom components of (10.1.29), that is, $V_{1,2}(z) = \big[V_+(z)\pm V_-(z)\big]/\sqrt{2}$:

$$\begin{aligned} V_1(z) &= \frac{e^{-j\beta_+ z}+\Gamma_{L+}e^{-2j\beta_+ l}e^{j\beta_+ z}}{1-\Gamma_{G+}\Gamma_{L+}e^{-2j\beta_+ l}}V_+ + \frac{e^{-j\beta_- z}+\Gamma_{L-}e^{-2j\beta_- l}e^{j\beta_- z}}{1-\Gamma_{G-}\Gamma_{L-}e^{-2j\beta_- l}}V_- \\ V_2(z) &= \frac{e^{-j\beta_+ z}+\Gamma_{L+}e^{-2j\beta_+ l}e^{j\beta_+ z}}{1-\Gamma_{G+}\Gamma_{L+}e^{-2j\beta_+ l}}V_+ - \frac{e^{-j\beta_- z}+\Gamma_{L-}e^{-2j\beta_- l}e^{j\beta_- z}}{1-\Gamma_{G-}\Gamma_{L-}e^{-2j\beta_- l}}V_- \end{aligned} \tag{10.1.31}$$

where we defined:

$$V_\pm = \left(\frac{Z_\pm}{Z_\pm + Z_G}\right)\frac{V_{G\pm}}{\sqrt{2}} = \frac{1}{4}(1-\Gamma_{G\pm})(V_{G1}\pm V_{G2}) \tag{10.1.32}$$

The parameters $\beta_\pm, Z_\pm$ are obtained using the rules of Eq. (10.1.15). From Eq. (10.1.12), we find the eigenvalues of the matrices $F \pm G$:

$$\begin{aligned} (F+G)_\pm &= \beta \pm (\kappa+\chi) = \beta\left(1 \pm \frac{L_m}{L_0}\right) = \omega\frac{1}{Z_0}(L_0 \pm L_m) \\ (F-G)_\pm &= \beta \pm (\kappa-\chi) = \beta\left(1 \mp \frac{C_m}{C_0}\right) = \omega Z_0(C_0 \mp C_m) \end{aligned}$$

Then, it follows that:

$$\begin{aligned} \beta_+ &= \sqrt{(F+G)_+(F-G)_+} = \omega\sqrt{(L_0+L_m)(C_0-C_m)} \\ \beta_- &= \sqrt{(F+G)_-(F-G)_-} = \omega\sqrt{(L_0-L_m)(C_0+C_m)} \end{aligned} \tag{10.1.33}$$

$$\begin{aligned} Z_+ &= Z_0\sqrt{\frac{(F+G)_+}{(F-G)_+}} = \sqrt{\frac{L_0+L_m}{C_0-C_m}} \\ Z_- &= Z_0\sqrt{\frac{(F+G)_-}{(F-G)_-}} = \sqrt{\frac{L_0-L_m}{C_0+C_m}} \end{aligned} \tag{10.1.34}$$

Thus, the coupled system acts as two uncoupled lines with wavenumbers and characteristic impedances $\beta_\pm, Z_\pm$, propagation speeds $v_\pm = 1/\sqrt{(L_0\pm L_m)(C_0\mp C_m)}$, and propagation delays $T_\pm = l/v_\pm$. The even mode is energized when $V_{G2} = V_{G1}$, or, $V_{G+} \neq 0, V_{G-} = 0$, and the odd mode, when $V_{G2} = -V_{G1}$, or, $V_{G+} = 0, V_{G-} \neq 0$.

When the coupled lines are immersed in a *homogeneous* medium, such as two parallel wires in air over a ground plane, then the propagation speeds must be equal to the speed of light within this medium [496], that is, $v_+ = v_- = 1/\sqrt{\mu\epsilon}$. This requires:

$$\begin{aligned} (L_0+L_m)(C_0-C_m) &= \mu\epsilon \\ (L_0-L_m)(C_0+C_m) &= \mu\epsilon \end{aligned} \quad\Rightarrow\quad \begin{aligned} L_0 &= \frac{\mu\epsilon C_0}{C_0^2 - C_m^2} \\ L_m &= \frac{\mu\epsilon C_m}{C_0^2 - C_m^2} \end{aligned} \tag{10.1.35}$$

Therefore, $L_m/L_0 = C_m/C_0$, or, equivalently, $\kappa = 0$. On the other hand, in an *inhomogeneous* medium, such as for the case of the microstrip lines shown in Fig. 10.1.1, the propagation speeds may be different, $v_+ \neq v_-$, and hence $T_+ \neq T_-$.

10.2 Crosstalk Between Lines

When only line-1 is energized, that is, $V_{G1}\neq 0, V_{G2}=0$, the coupling between the lines induces a propagating wave in line-2, referred to as crosstalk, which also has some minor influence back on line-1. The *near-end* and *far-end* crosstalk are the values of $V_2(z)$ at $z=0$ and $z=l$, respectively. Setting $V_{G2}=0$ in (10.1.32), we have from (10.1.31):

$$\begin{aligned} V_2(0) &= \frac{1}{2}\frac{(1-\Gamma_{G+})(1+\Gamma_{L+}\zeta_+^{-2})}{1-\Gamma_{G+}\Gamma_{L+}\zeta_+^{-2}}V - \frac{1}{2}\frac{(1-\Gamma_{G-})(1+\Gamma_{L-}\zeta_-^{-1})}{1-\Gamma_{G-}\Gamma_{L-}\zeta^{-2}}V \\ V_2(l) &= \frac{1}{2}\frac{\zeta_+^{-1}(1-\Gamma_{G+})(1+\Gamma_{L+})}{1-\Gamma_{G+}\Gamma_{L+}\zeta_+^{-2}}V - \frac{1}{2}\frac{\zeta_-^{-1}(1-\Gamma_{G-})(1+\Gamma_{L-})}{1-\Gamma_{G-}\Gamma_{L-}\zeta_-^{-2}}V \end{aligned} \tag{10.2.1}$$

where we defined $V=V_{G1}/2$ and introduced the z-transform delay variables $\zeta_\pm = e^{j\omega T_\pm}=e^{j\beta_\pm l}$. Assuming purely resistive termination impedances Z_G, Z_L, we may use Eq. (9.15.15) to obtain the corresponding time-domain responses:

$$\begin{aligned} V_2(0,t) &= \frac{1}{2}(1-\Gamma_{G+})\left[V(t)+\left(1+\frac{1}{\Gamma_{G+}}\right)\sum_{m=1}^{\infty}(\Gamma_{G+}\Gamma_{L+})^m V(t-2mT_+)\right] \\ &\quad - \frac{1}{2}(1-\Gamma_{G-})\left[V(t)+\left(1+\frac{1}{\Gamma_{G-}}\right)\sum_{m=1}^{\infty}(\Gamma_{G-}\Gamma_{L-})^m V(t-2mT_-)\right] \\ V_2(l,t) &= \frac{1}{2}(1-\Gamma_{G+})(1+\Gamma_{L+})\sum_{m=0}^{\infty}(\Gamma_{G+}\Gamma_{L+})^m V(t-2mT_+-T_+) \\ &\quad - \frac{1}{2}(1-\Gamma_{G-})(1+\Gamma_{L-})\sum_{m=0}^{\infty}(\Gamma_{G-}\Gamma_{L-})^m V(t-2mT_--T_-) \end{aligned} \tag{10.2.2}$$

where $V(t)=V_{G1}(t)/2$.† Because $Z_\pm\neq Z_0$, there will be multiple reflections even when the lines are matched to Z_0 at both ends. Setting $Z_G=Z_L=Z_0$, gives for the reflection coefficients (10.1.30):

$$\Gamma_{G\pm}=\Gamma_{L\pm}=\frac{Z_0-Z_\pm}{Z_0+Z_\pm}=-\Gamma_\pm \tag{10.2.3}$$

In this case, we find for the crosstalk signals:

$$\begin{aligned} V_2(0,t) &= \frac{1}{2}(1+\Gamma_+)\left[V(t)-(1-\Gamma_+)\sum_{m=1}^{\infty}\Gamma_+^{2m-1}V(t-2mT_+)\right] \\ &\quad - \frac{1}{2}(1+\Gamma_-)\left[V(t)-(1-\Gamma_-)\sum_{m=1}^{\infty}\Gamma_-^{2m-1}V(t-2mT_-)\right] \\ V_2(l,t) &= \frac{1}{2}(1-\Gamma_+^2)\sum_{m=0}^{\infty}\Gamma_+^{2m}V(t-2mT_+-T_+) \\ &\quad - \frac{1}{2}(1-\Gamma_-^2)\sum_{m=0}^{\infty}\Gamma_-^{2m}V(t-2mT_--T_-) \end{aligned} \tag{10.2.4}$$

† $V(t)$ is the signal that would exist on a matched line-1 in the absence of line-2, $V=Z_0V_{G1}/(Z_0+Z_G)=V_{G1}/2$, provided $Z_G=Z_0$.

Similarly, the near-end and far-end signals on the driven line are found by adding, instead of subtracting, the even- and odd-mode terms:

$$\begin{aligned} V_1(0,t) &= \frac{1}{2}(1+\Gamma_+)\left[V(t)-(1-\Gamma_+)\sum_{m=1}^{\infty}\Gamma_+^{2m-1}V(t-2mT_+)\right] \\ &\quad + \frac{1}{2}(1+\Gamma_-)\left[V(t)-(1-\Gamma_-)\sum_{m=1}^{\infty}\Gamma_-^{2m-1}V(t-2mT_-)\right] \\ V_1(l,t) &= \frac{1}{2}(1-\Gamma_+^2)\sum_{m=0}^{\infty}\Gamma_+^{2m}V(t-2mT_+-T_+) \\ &\quad + \frac{1}{2}(1-\Gamma_-^2)\sum_{m=0}^{\infty}\Gamma_-^{2m}V(t-2mT_--T_-) \end{aligned} \tag{10.2.5}$$

These expressions simplify drastically if we assume weak coupling. It is straightforward to verify that to *first-order* in the parameters $L_m/L_0, C_m/C_0$, or equivalently, to first-order in κ,χ, we have the approximations:

$$\begin{aligned} &\beta_\pm = \beta\pm\Delta\beta = \beta\pm\kappa\,, \quad Z_\pm = Z_0\pm\Delta Z = Z_0\pm Z_0\frac{\chi}{\beta}\,, \quad \nu_\pm = \nu_0\mp\nu_0\frac{\kappa}{\beta} \\ &\Gamma_\pm = 0\pm\Delta\Gamma = \pm\frac{\chi}{2\beta}\,, \quad T_\pm = T\pm\Delta T = T\pm T\frac{\kappa}{\beta} \end{aligned} \tag{10.2.6}$$

where $T=l/\nu_0$. Because the $\Gamma_\pm$s are already first-order, the multiple reflection terms in the above summations are a second-order effect, and only the lowest terms will contribute, that is, the term $m=1$ for the near-end, and $m=0$ for the far end. Then,

$$\begin{aligned} V_2(0,t) &= \frac{1}{2}(\Gamma_+-\Gamma_-)V(t)-\frac{1}{2}\left[\Gamma_+V(t-2T_+)-\Gamma_-V(t-2T_-)\right] \\ V_2(l,t) &= \frac{1}{2}\left[V(t-T_+)-V(t-T_-)\right] \end{aligned}$$

Using a Taylor series expansion and (10.2.6), we have to first-order:

$$\begin{aligned} V(t-2T_\pm) &= V(t-2T\mp\Delta T)\simeq V(t-2T)\mp(\Delta T)\dot{V}(t-2T)\,, \quad \dot{V}=\frac{dV}{dt} \\ V(t-T_\pm) &= V(t-T\mp\Delta T)\simeq V(t-T)\mp(\Delta T)\dot{V}(t-T) \end{aligned}$$

Therefore, $\Gamma_\pm V(t-2T_\pm)=\Gamma_\pm\left[V(t-2T)\mp(\Delta T)\dot{V}\right]\simeq\Gamma_\pm V(t-2T)$, where we ignored the second-order terms $\Gamma_\pm(\Delta T)\dot{V}$. It follows that:

$$\begin{aligned} V_2(0,t) &= \frac{1}{2}(\Gamma_+-\Gamma_-)\left[V(t)-V(t-2T)\right] = (\Delta\Gamma)\left[V(t)-V(t-2T)\right] \\ V_2(l,t) &= \frac{1}{2}\left[V(t-T)-(\Delta T)\dot{V}-V(t-T)-(\Delta T)\dot{V}\right] = -(\Delta T)\frac{dV(t-T)}{dt} \end{aligned}$$

These can be written in the commonly used form:

$$\boxed{\begin{aligned} V_2(0,t) &= K_b\left[V(t)-V(t-2T)\right] \\ V_2(l,t) &= K_f\frac{dV(t-T)}{dt} \end{aligned}} \quad \text{(near- and far-end crosstalk)} \tag{10.2.7}$$

where K_b, K_f are known as the backward and forward crosstalk coefficients:

$$K_b = \frac{\chi}{2\beta} = \frac{v_0}{4}\left(\frac{L_m}{Z_0} + C_m Z_0\right), \quad K_f = -T\frac{\kappa}{\beta} = -\frac{v_0 T}{2}\left(\frac{L_m}{Z_0} - C_m Z_0\right) \tag{10.2.8}$$

where we may replace $l = v_0 T$. The same approximations give for line-1, $V_1(0,t) = V(t)$ and $V_1(l,t) = V(t-T)$. Thus, to first-order, line-2 does not act back to disturb line-1.

Example 10.2.1: Fig. 10.2.1 shows the signals $V_1(0,t)$, $V_1(l,t)$, $V_2(0,t)$, $V_2(l,t)$ for a pair of coupled lines matched at both ends. The uncoupled line impedance was $Z_0 = 50\ \Omega$.

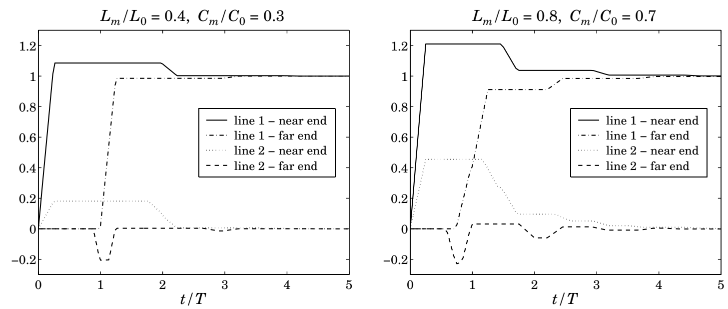


Fig. 10.2.1 Near- and far-end crosstalk signals on lines 1 and 2.

For the left graph, we chose $L_m/L_0 = 0.4$, $C_m/C_0 = 0.3$, which results in the even and odd mode parameters (using the exact formulas):

$$Z_+ = 70.71\ \Omega, \quad Z_- = 33.97\ \Omega, \quad v_+ = 1.01v_0, \quad v_- = 1.13v_0$$
$$\Gamma_+ = -0.17, \quad \Gamma_- = 0.19, \quad T_+ = 0.99T, \quad T_- = 0.88T, \quad K_b = 0.175, \quad K_f = 0.05$$

The right graph corresponds to $L_m/L_0 = 0.8$, $C_m/C_0 = 0.7$, with parameters:

$$Z_+ = 122.47\ \Omega, \quad Z_- = 17.15\ \Omega, \quad v_+ = 1.36v_0, \quad v_- = 1.71v_0$$
$$\Gamma_+ = -0.42, \quad \Gamma_- = 0.49, \quad T_+ = 0.73T, \quad T_- = 0.58T, \quad K_b = 0.375, \quad K_f = 0.05$$

The generator input to line-1 was a rising step with rise-time $t_r = T/4$, that is,

$$V(t) = \frac{1}{2}V_{G1}(t) = \frac{t}{t_r}\big[u(t) - u(t-t_r)\big] + u(t-t_r)$$

The weak-coupling approximations are more closely satisfied for the left case. Eqs. (10.2.7) predict for $V_2(0,t)$ a trapezoidal pulse of duration $2T$ and height K_b, and for $V_2(l,t)$, a rectangular pulse of width t_r and height $K_f/t_r = -0.2$ starting at $t = T$:

$$V_2(l,t) = K_f\frac{dV(t-T)}{dt} = \frac{K_f}{t_r}\big[u(t-T) - u(t-T-t_r)\big]$$

These predictions are approximately correct as can be seen in the figure. The approximation predicts also that $V_1(0,t) = V(t)$ and $V_1(l,t) = V(t-T)$, which are not quite true—the effect of line-2 on line-1 cannot be ignored completely.

The interaction between the two lines is seen better in the MATLAB movie `xtalkmovie.m`, which plots the waves $V_1(z,t)$ and $V_2(z,t)$ as they propagate to and get reflected from their respective loads, and compares them to the uncoupled case $V_0(z,t) = V(t - z/v_0)$. The waves $V_{1,2}(z,t)$ are computed by the same method as for the movie `pulsemovie.m` of Example 9.15.1, applied separately to the even and odd modes. □

10.3 Weakly Coupled Lines with Arbitrary Terminations

The even-odd mode decomposition can be carried out only in the case of *identical* lines both of which have the *same* load and generator impedances. The case of arbitrary terminations has been solved in closed form only for homogeneous media [493,496]. It has also been solved for arbitrary media under the weak coupling assumption [503].

Following [503], we solve the general equations (10.1.7)–(10.1.9) for weakly coupled lines assuming arbitrary terminating impedances Z_{Li}, Z_{Gi}, with reflection coefficients:

$$\Gamma_{Li} = \frac{Z_{Li} - Z_i}{Z_{Li} + Z_i}, \quad \Gamma_{Gi} = \frac{Z_{Gi} - Z_i}{Z_{Gi} + Z_i}, \quad i = 1, 2 \tag{10.3.1}$$

Working with the forward and backward waves, we write Eq. (10.1.7) as the 4×4 matrix equation:

$$\frac{d\mathbf{c}}{dz} = -jM\mathbf{c}, \qquad \mathbf{c} = \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix}, \quad M = \begin{bmatrix} \beta_1 & \kappa & 0 & -\chi \\ \kappa & \beta_2 & -\chi & 0 \\ 0 & \chi & -\beta_1 & -\kappa \\ \chi & 0 & -\kappa & -\beta_2 \end{bmatrix}$$

The weak coupling assumption consists of ignoring the coupling of a_1, b_1 on a_2, b_2. This amounts to approximating the above linear system by:

$$\frac{d\mathbf{c}}{dz} = -j\hat{M}\mathbf{c}, \qquad \hat{M} = \begin{bmatrix} \beta_1 & 0 & 0 & 0 \\ \kappa & \beta_2 & -\chi & 0 \\ 0 & 0 & -\beta_1 & 0 \\ \chi & 0 & -\kappa & -\beta_2 \end{bmatrix} \tag{10.3.2}$$

Its solution is given by $\mathbf{c}(z) = e^{-j\hat{M}z}\mathbf{c}(0)$, where the transition matrix $e^{-j\hat{M}z}$ can be expressed in closed form as follows:

$$e^{-j\hat{M}z} = \begin{bmatrix} e^{-j\beta_1 z} & 0 & 0 & 0 \\ \hat{\kappa}(e^{-j\beta_1 z} - e^{-j\beta_2 z}) & e^{-j\beta_2 z} & \hat{\chi}(e^{j\beta_1 z} - e^{-j\beta_2 z}) & 0 \\ 0 & 0 & e^{j\beta_1 z} & 0 \\ \hat{\chi}(e^{-j\beta_1 z} - e^{j\beta_2 z}) & 0 & \hat{\kappa}(e^{j\beta_1 z} - e^{j\beta_2 z}) & e^{j\beta_2 z} \end{bmatrix}, \quad \begin{aligned} \hat{\kappa} &= \frac{\kappa}{\beta_1 - \beta_2} \\ \hat{\chi} &= \frac{\chi}{\beta_1 + \beta_2} \end{aligned}$$

The transition matrix $e^{-j\hat{M}l}$ may be written in terms of the z-domain delay variables $\zeta_i = e^{j\beta_i l} = e^{j\omega T_i}$, $i = 1, 2$, where T_i are the one-way travel times along the lines, that is, $T_i = l/v_i$. Then, we find:

$$\begin{bmatrix} a_1(l) \\ a_2(l) \\ b_1(l) \\ b_2(l) \end{bmatrix} = \begin{bmatrix} \zeta_1^{-1} & 0 & 0 & 0 \\ \hat{\kappa}(\zeta_1^{-1} - \zeta_2^{-1}) & \zeta_2^{-1} & \hat{\chi}(\zeta_1 - \zeta_2^{-1}) & 0 \\ 0 & 0 & \zeta_1 & 0 \\ \hat{\chi}(\zeta_1^{-1} - \zeta_2) & 0 & \hat{\kappa}(\zeta_1 - \zeta_2) & \zeta_2 \end{bmatrix} \begin{bmatrix} a_1(0) \\ a_2(0) \\ b_1(0) \\ b_2(0) \end{bmatrix} \tag{10.3.3}$$

These must be appended by the appropriate terminating conditions. Assuming that only line-1 is driven, we have:

$$\begin{aligned} &V_1(0)+Z_{G1}I_1(0)=V_{G1}\,, && V_1(l)=Z_{L1}I_1(l)\\ &V_2(0)+Z_{G2}I_2(0)=0\,, && V_2(l)=Z_{L2}I_2(l) \end{aligned}$$

which can be written in terms of the $\mathbf{a},\mathbf{b}$ waves:

$$\begin{aligned} &a_1(0)-\Gamma_{G1}b_1(0)=U_1\,, && b_1(l)=\Gamma_{L1}a_1(l)\\ &a_2(0)-\Gamma_{G2}b_2(0)=0\,, && b_2(l)=\Gamma_{L2}a_2(l) \end{aligned}\;,\quad U_1=\sqrt{\frac{2}{Z_1}}(1-\Gamma_{G1})\frac{V_{G1}}{2} \tag{10.3.4}$$

Eqs. (10.3.3) and (10.3.4) provide a set of eight equations in eight unknowns. Once these are solved, the near- and far-end voltages may be determined. For line-1, we find:

$$\begin{aligned} V_1(0)&=\sqrt{\frac{Z_1}{2}}\big[a_1(0)+b_1(0)\big]=\frac{1+\Gamma_{L1}\zeta_1^{-2}}{1-\Gamma_{G1}\Gamma_{L1}\zeta_1^{-2}}\,V\\ V_1(l)&=\sqrt{\frac{Z_1}{2}}\big[a_1(l)+b_1(l)\big]=\frac{\zeta_1^{-1}(1+\Gamma_{L1})}{1-\Gamma_{G1}\Gamma_{L1}\zeta_1^{-2}}\,V \end{aligned} \tag{10.3.5}$$

where $V=(1-\Gamma_{G1})V_{G1}/2=Z_1V_{G1}/(Z_1+Z_{G1})$. For line-2, we have:

$$\begin{aligned} V_2(0)&=\frac{\bar\kappa(\zeta_1^{-1}-\zeta_2^{-1})(\Gamma_{L1}\zeta_1^{-1}+\Gamma_{L2}\zeta_2^{-1})+\bar\chi(1-\zeta_1^{-1}\zeta_2^{-1})(1+\Gamma_{L1}\Gamma_{L2}\zeta_1^{-1}\zeta_2^{-1})}{(1-\Gamma_{G1}\Gamma_{L1}\zeta_1^{-2})(1-\Gamma_{G2}\Gamma_{L2}\zeta_2^{-2})}\,V_{20}\\ V_2(l)&=\frac{\bar\kappa(\zeta_1^{-1}-\zeta_2^{-1})(1+\Gamma_{L1}\Gamma_{G2}\zeta_1^{-1}\zeta_2^{-1})+\bar\chi(1-\zeta_1^{-1}\zeta_2^{-1})(\Gamma_{L1}\zeta_1^{-1}+\Gamma_{G2}\zeta_2^{-1})}{(1-\Gamma_{G1}\Gamma_{L1}\zeta_1^{-2})(1-\Gamma_{G2}\Gamma_{L2}\zeta_2^{-2})}\,V_{2l} \end{aligned} \tag{10.3.6}$$

where $V_{20}=(1+\Gamma_{G2})V=(1+\Gamma_{G2})(1-\Gamma_{G1})V_{G1}/2$ and $V_{2l}=(1+\Gamma_{L2})V$, and we defined $\bar\kappa,\bar\chi$ by:

$$\begin{aligned} \bar\kappa&=\sqrt{\frac{Z_2}{Z_1}}\,\hat\kappa=\sqrt{\frac{Z_2}{Z_1}}\,\frac{\kappa}{\beta_1-\beta_2}=\frac{\omega}{\beta_1-\beta_2}\,\frac{1}{2}\left(\frac{L_m}{Z_1}-C_mZ_2\right)\\ \bar\chi&=\sqrt{\frac{Z_2}{Z_1}}\,\hat\chi=\sqrt{\frac{Z_2}{Z_1}}\,\frac{\chi}{\beta_1+\beta_2}=\frac{\omega}{\beta_1+\beta_2}\,\frac{1}{2}\left(\frac{L_m}{Z_1}+C_mZ_2\right) \end{aligned} \tag{10.3.7}$$

In the case of identical lines with $Z_1=Z_2=Z_0$ and $\beta_1=\beta_2=\beta=\omega/v_0$, we must take the limit:

$$\lim_{\beta_2\to\beta_1}\frac{e^{-j\beta_1 l}-e^{-j\beta_2 l}}{\beta_1-\beta_2}=\frac{d}{d\beta_1}e^{-j\beta_1 l}=-jle^{-j\beta_1 l}$$

Then, we obtain:

$$\begin{aligned} &\bar\kappa(\zeta_1^{-1}-\zeta_2^{-1})\to j\omega K_f e^{-j\beta l}=-j\omega\,\frac{l}{2}\left(\frac{L_m}{Z_0}-C_mZ_0\right)e^{-j\beta l}\\ &\bar\chi\to K_b=\frac{v_0}{4}\left(\frac{L_m}{Z_0}+C_mZ_0\right) \end{aligned} \tag{10.3.8}$$

where K_f,K_b were defined in (10.2.8). Setting $\zeta_1=\zeta_2=\zeta=e^{j\beta l}=e^{j\omega T}$, we obtain the crosstalk signals:

$$\begin{aligned} V_2(0)&=\frac{j\omega K_f(\Gamma_{L1}+\Gamma_{L2})\zeta^{-2}+K_b(1-\zeta^{-2})(1+\Gamma_{L1}\Gamma_{L2}\zeta^{-2})}{(1-\Gamma_{G1}\Gamma_{L1}\zeta^{-2})(1-\Gamma_{G2}\Gamma_{L2}\zeta^{-2})}\,V_{20}\\ V_2(l)&=\frac{j\omega K_f(1+\Gamma_{L1}\Gamma_{G2}\zeta^{-2})\zeta^{-1}+K_b(1-\zeta^{-2})(\Gamma_{L1}+\Gamma_{G2})\zeta^{-1}}{(1-\Gamma_{G1}\Gamma_{L1}\zeta^{-2})(1-\Gamma_{G2}\Gamma_{L2}\zeta^{-2})}\,V_{2l} \end{aligned} \tag{10.3.9}$$

The corresponding time-domain signals will involve the double multiple reflections arising from the denominators. However, if we assume the each line is matched in at least one of its ends, so that $\Gamma_{G1}\Gamma_{L1}=\Gamma_{G2}\Gamma_{L2}=0$, then the denominators can be eliminated. Replacing $j\omega$ by the time-derivative d/dt and each factor ζ^{-1} by a delay by T, we obtain:

$$\begin{aligned} V_2(0,t)&=K_f(\Gamma_{L1}+\Gamma_{L2}+\Gamma_{L1}\Gamma_{G2})\dot V(t-2T)\\ &\quad+K_b(1+\Gamma_{G2})\big[V(t)-V(t-2T)\big]+K_b\Gamma_{L1}\Gamma_{L2}\big[V(t-2T)-V(t-4T)\big]\\ V_2(l,t)&=K_f\big[(1+\Gamma_{L2})\dot V(t-T)+\Gamma_{L1}\Gamma_{G2}\dot V(t-3T)\big]\\ &\quad+K_b(\Gamma_{L1}+\Gamma_{G2}+\Gamma_{L1}\Gamma_{L2})\big[V(t-T)-V(t-3T)\big] \end{aligned} \tag{10.3.10}$$

where $V(t)=(1-\Gamma_{G1})V_{G1}(t)/2$, and we used the property $\Gamma_{G2}\Gamma_{L2}=0$ to simplify the expressions. Eqs. (10.3.10) reduce to (10.2.7) when the lines are matched at both ends.

10.4 Coupled-Mode Theory

In its simplest form, *coupled-mode* or *coupled-wave theory* provides a paradigm for the interaction between two waves and the exchange of energy from one to the other as they propagate. Reviews and earlier literature may be found in Refs. [504–525], see also [353–372] for the relationship to fiber Bragg gratings and distributed feedback lasers.

There are several mechanical and electrical analogs of coupled-mode theory, such as a pair of coupled pendula, or two masses at the ends of two springs with a third spring connecting the two, or two LC circuits with a coupling capacitor between them. In these examples, the exchange of energy is taking place over time instead of over space.

Coupled-wave theory is inherently directional. If two forward-moving waves are strongly coupled, then their interactions with the corresponding backward waves may be ignored. Similarly, if a forward- and a backward-moving wave are strongly coupled, then their interactions with the corresponding oppositely moving waves may be ignored. Fig. 10.4.1 depicts these two cases of *co-directional* and *contra-directional* coupling.

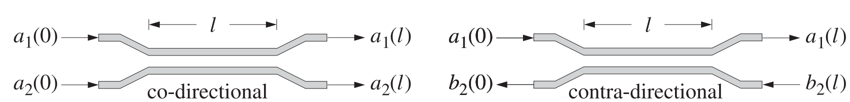


Fig. 10.4.1 Directional Couplers.

Eqs. (10.1.7) form the basis of coupled-mode theory. In the co-directional case, if we assume that there are only forward waves at $z=0$, that is, $\mathbf{a}(0)\neq 0$ and $\mathbf{b}(0)=0$, then it may shown that the effect of the backward waves on the forward ones becomes a second-order effect in the coupling constants, and therefore, it may be ignored. To see this, we solve the second of Eqs. (10.1.7) for **b** in terms of **a**, assuming zero initial conditions, and substitute it in the first:

$$\mathbf{b}(z)=-j\int_0^z e^{jF(z-z')}G\,\mathbf{a}(z')\,dz' \quad\Rightarrow\quad \frac{d\mathbf{a}}{dz}=-jF\,\mathbf{a}+\int_0^z Ge^{jF(z-z')}G\,\mathbf{a}(z')\,dz'$$

The second term is second-order in G, or in the coupling constant χ. Ignoring this term, we obtain the standard equations describing a co-directional coupler:

$$\frac{d\mathbf{a}}{dz}=-jF\,\mathbf{a} \quad\Rightarrow\quad \frac{d}{dz}\begin{bmatrix}a_1\\a_2\end{bmatrix}=-j\begin{bmatrix}\beta_1 & \kappa\\ \kappa & \beta_2\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix} \tag{10.4.1}$$

For the contra-directional case, a similar argument that assumes the initial conditions $a_2(0)=b_1(0)=0$ gives the following approximation that couples the a_1 and b_2 waves:

$$\frac{d}{dz}\begin{bmatrix}a_1\\b_2\end{bmatrix}=-j\begin{bmatrix}\beta_1 & -\chi\\ \chi & -\beta_2\end{bmatrix}\begin{bmatrix}a_1\\b_2\end{bmatrix} \tag{10.4.2}$$

The conserved powers are in the two cases:

$$P=|a_1|^2+|a_2|^2\,,\qquad P=|a_1|^2-|b_2|^2 \tag{10.4.3}$$

The solution of Eq. (10.4.1) is obtained with the help of the transition matrix e^{-jFz}:

$$e^{-jFz}=e^{-j\beta z}\begin{bmatrix}\cos\sigma z-j\dfrac{\delta}{\sigma}\sin\sigma z & -j\dfrac{\kappa}{\sigma}\sin\sigma z\\ -j\dfrac{\kappa}{\sigma}\sin\sigma z & \cos\sigma z+j\dfrac{\delta}{\sigma}\sin\sigma z\end{bmatrix} \tag{10.4.4}$$

where

$$\beta=\frac{\beta_1+\beta_2}{2}\,,\quad \delta=\frac{\beta_1-\beta_2}{2}\,,\quad \sigma=\sqrt{\delta^2+\kappa^2} \tag{10.4.5}$$

Thus, the solution of (10.4.1) is:

$$\begin{bmatrix}a_1(z)\\a_2(z)\end{bmatrix}=e^{-j\beta z}\begin{bmatrix}\cos\sigma z-j\dfrac{\delta}{\sigma}\sin\sigma z & -j\dfrac{\kappa}{\sigma}\sin\sigma z\\ -j\dfrac{\kappa}{\sigma}\sin\sigma z & \cos\sigma z-j\dfrac{\delta}{\sigma}\sin\sigma z\end{bmatrix}\begin{bmatrix}a_1(0)\\a_2(0)\end{bmatrix} \tag{10.4.6}$$

Starting with initial conditions $a_1(0)=1$ and $a_2(0)=0$, the total initial power will be $P=|a_1(0)|^2+|a_2(0)|^2=1$. As the waves propagate along the z-direction, power is exchanged between lines 1 and 2 according to:

$$\begin{aligned}P_1(z)&=|a_1(z)|^2=\cos^2\sigma z+\frac{\delta^2}{\sigma^2}\sin^2\sigma z\\ P_2(z)&=|a_2(z)|^2=\frac{\kappa^2}{\sigma^2}\sin^2\sigma z=1-P_1(z)\end{aligned} \tag{10.4.7}$$

Fig. 10.4.2 shows the two cases for which $\delta/\kappa=0$ and $\delta/\kappa=0.5$. In both cases, maximum exchange of power occurs periodically at distances that are odd multiples of $z=\pi/2\sigma$. Complete power exchange occurs only in the case $\delta=0$, or equivalently, when $\beta_1=\beta_2$. In this case, we have $\sigma=\kappa$ and $P_1(z)=\cos^2\kappa z$, $P_2(z)=\sin^2\kappa z$.

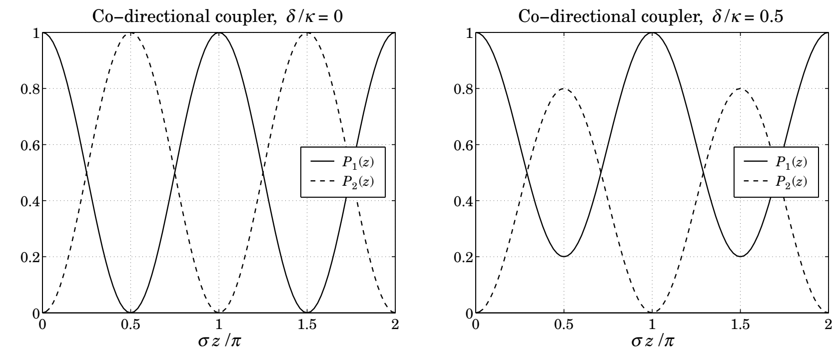


Fig. 10.4.2 Power exchange in co-directional couplers.

10.5 *Fiber Bragg Gratings*

As an example of contra-directional coupling, we consider the case of a *fiber Bragg grating* (FBG), that is, a fiber with a segment that has a periodically varying refractive index, as shown in Fig. 10.5.1.

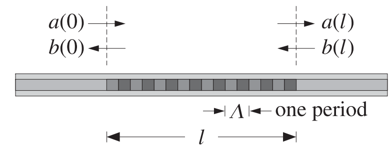


Fig. 10.5.1 Fiber Bragg grating.

The backward wave is generated by the reflection of a forward-moving wave incident on the interface from the left. The grating behaves very similarly to a periodic multilayer structure, such as a dielectric mirror at normal incidence, exhibiting high-reflectance bands. A simple model for an FBG is as follows [353–372]:

$$\frac{d}{dz}\begin{bmatrix}a(z)\\b(z)\end{bmatrix}=-j\begin{bmatrix}\beta & \kappa e^{-jKz}\\ -\kappa^* e^{jKz} & -\beta\end{bmatrix}\begin{bmatrix}a(z)\\b(z)\end{bmatrix} \tag{10.5.1}$$

where $K=2\pi/\Lambda$ is the Bloch wavenumber, Λ is the period, and $a(z),b(z)$ represent the forward and backward waves. The following transformation removes the phase factor e^{-jKz} from the coupling constant:

$$\begin{bmatrix}A(z)\\B(z)\end{bmatrix}=\begin{bmatrix}e^{jKz/2} & 0\\ 0 & e^{-jKz/2}\end{bmatrix}\begin{bmatrix}a(z)\\b(z)\end{bmatrix}=\begin{bmatrix}e^{jKz/2}a(z)\\ e^{-jKz/2}b(z)\end{bmatrix} \tag{10.5.2}$$

$$\frac{d}{dz}\begin{bmatrix}A(z)\\B(z)\end{bmatrix}=-j\begin{bmatrix}\delta & \kappa\\ -\kappa^* & -\delta\end{bmatrix}\begin{bmatrix}A(z)\\B(z)\end{bmatrix} \tag{10.5.3}$$

where $\delta = \beta - K/2$ is referred to as a *detuning parameter*. The conserved power is given by $P(z) = |a(z)|^2 - |b(z)|^2$. The fields at $z = 0$ are related to those at $z = l$ by:

$$\begin{bmatrix} A(0) \\ B(0) \end{bmatrix} = e^{jFl} \begin{bmatrix} A(l) \\ B(l) \end{bmatrix}, \quad \text{with} \quad F = \begin{bmatrix} \delta & \kappa \\ -\kappa^* & -\delta \end{bmatrix} \tag{10.5.4}$$

The transfer matrix e^{jFl} is given by:

$$e^{jFl} = \begin{bmatrix} \cos\sigma l + j\dfrac{\delta}{\sigma}\sin\sigma l & j\dfrac{\kappa}{\sigma}\sin\sigma l \\ -j\dfrac{\kappa^*}{\sigma}\sin\sigma l & \cos\sigma l - j\dfrac{\delta}{\sigma}\sin\sigma l \end{bmatrix} \equiv \begin{bmatrix} U_{11} & U_{12} \\ U_{12}^* & U_{11}^* \end{bmatrix} \tag{10.5.5}$$

where $\sigma = \sqrt{\delta^2 - |\kappa|^2}$. If $|\delta| < |\kappa|$, then σ becomes imaginary. In this case, it is more convenient to express the transfer matrix in terms of the quantity $\gamma = \sqrt{|\kappa|^2 - \delta^2}$:

$$e^{jFl} = \begin{bmatrix} \cosh\gamma l + j\dfrac{\delta}{\gamma}\sinh\gamma l & j\dfrac{\kappa}{\gamma}\sinh\gamma l \\ -j\dfrac{\kappa^*}{\gamma}\sinh\gamma l & \cosh\gamma l - j\dfrac{\delta}{\gamma}\sinh\gamma l \end{bmatrix} \tag{10.5.6}$$

The transfer matrix has unit determinant, which implies that $|U_{11}|^2 - |U_{12}|^2 = 1$. Using this property, we may rearrange (10.5.4) into its *scattering matrix* form that relates the outgoing fields to the incoming ones:

$$\begin{bmatrix} B(0) \\ A(l) \end{bmatrix} = \begin{bmatrix} \Gamma & T \\ T & \Gamma' \end{bmatrix} \begin{bmatrix} A(0) \\ B(l) \end{bmatrix}, \quad \Gamma = \frac{U_{12}^*}{U_{11}}, \quad \Gamma' = -\frac{U_{12}}{U_{11}}, \quad T = \frac{1}{U_{11}} \tag{10.5.7}$$

where Γ, Γ' are the *reflection* coefficients from the left and right, respectively, and T is the transmission coefficient. We have explicitly,

$$\Gamma = \frac{-j\dfrac{\kappa^*}{\sigma}\sin\sigma l}{\cos\sigma l + j\dfrac{\delta}{\sigma}\sin\sigma l}, \quad T = \frac{1}{\cos\sigma l + j\dfrac{\delta}{\sigma}\sin\sigma l} \tag{10.5.8}$$

If there is only an incident wave from the left, that is, $A(0) \neq 0$ and $B(l) = 0$, then (10.5.7) implies that $B(0) = \Gamma A(0)$ and $A(l) = TA(0)$.

A consequence of power conservation, $|A(0)|^2 - |B(0)|^2 = |A(l)|^2 - |B(l)|^2$, is the unitarity of the scattering matrix, which implies the property $|\Gamma|^2 + |T|^2 = 1$. The reflectance $|\Gamma|^2$ may be expressed in the following two forms, the first being appropriate when $|\delta| \geq |\kappa|$, and the second when $|\delta| \leq |\kappa|$:

$$|\Gamma|^2 = 1 - |T|^2 = \frac{|\kappa|^2 \sin^2\sigma l}{\sigma^2\cos^2\sigma l + \delta^2\sin^2\sigma l} = \frac{|\kappa|^2\sinh^2\gamma l}{\gamma^2\cosh^2\gamma l + \delta^2\sinh^2\gamma l} \tag{10.5.9}$$

Fig. 10.5.2 shows $|\Gamma|^2$ as a function of δ. The high-reflectance band corresponds to the range $|\delta| \leq |\kappa|$. The left graph has $\kappa l = 3$ and the right one $\kappa l = 6$.

As κl increases, the reflection band becomes sharper. The asymptotic width of the band is $-|\kappa| \leq \delta \leq |\kappa|$. For any finite value of κl, the maximum reflectance achieved

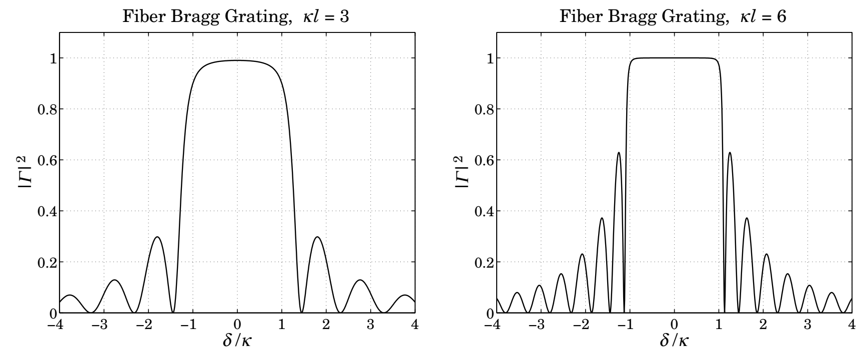


Fig. 10.5.2 Reflectance of fiber Bragg gratings.

at the center of the band, $\delta = 0$, is given by $|\Gamma|^2_{\max} = \tanh^2|\kappa l|$. The reflectance at the asymptotic band edges is given by:

$$|\Gamma|^2 = \frac{|\kappa l|^2}{1 + |\kappa l|^2}, \quad \text{at} \quad \delta = \pm|\kappa|$$

The zeros of the reflectance correspond to $\sin\sigma l = 0$, or, $\sigma = m\pi/l$, which gives $\delta = \pm\sqrt{|\kappa|^2 + (m\pi/l)^2}$, where m is a non-zero integer.

The *Bragg wavelength* λ_B is the wavelength at the center of the reflecting band, that is, corresponding to $\delta = 0$, or, $\beta = K/2$, or $\lambda_B = 2\pi/\beta = 4\pi/K = 2\Lambda$.

By concatenating two identical FBGs separated by a "spacer" of length $d = \lambda_B/4 = \Lambda/2$, we obtain a *quarter-wave phase-shifted FBG*, which has a narrow transmission window centered at $\delta = 0$. Fig. 10.5.3 depicts such a compound grating. Within the spacer, the A, B waves propagate with wavenumber β as though they are uncoupled.

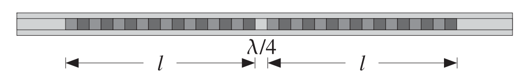


Fig. 10.5.3 Quarter-wave phase-shifted fiber Bragg grating.

The compound transfer matrix is obtained by multiplying the transfer matrices of the two FBGs and the spacer: $V = U_{\text{FBG}} U_{\text{spacer}} U_{\text{FBG}}$, or, explicitly:

$$\begin{bmatrix} V_{11} & V_{12} \\ V_{12}^* & V_{11}^* \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{12}^* & U_{11}^* \end{bmatrix} \begin{bmatrix} e^{j\beta d} & 0 \\ 0 & e^{-j\beta d} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{12}^* & U_{11}^* \end{bmatrix} \tag{10.5.10}$$

where the U_{ij} are given in Eq. (10.5.5). It follows that the matrix elements of V are:

$$V_{11} = U_{11}^2 e^{j\beta d} + |U_{12}|^2 e^{-j\beta d}, \quad V_{12} = U_{12}\left(U_{11}e^{j\beta d} + U_{11}^* e^{-j\beta d}\right) \tag{10.5.11}$$

The reflection coefficient of the compound grating will be:

$$\Gamma_{\text{comp}} = \frac{V_{12}^*}{V_{11}} = \frac{U_{12}\left(U_{11}e^{j\beta d} + U_{11}^* e^{-j\beta d}\right)}{U_{11}^2 e^{j\beta d} + |U_{12}|^2 e^{-j\beta d}} = \frac{\Gamma\left(T^* e^{j\beta d} + Te^{-j\beta d}\right)}{T^* e^{j\beta d} + |\Gamma|^2 Te^{-j\beta d}} \tag{10.5.12}$$

where we replaced $U_{12}^* = \Gamma/T$ and $U_{11} = 1/T$. Assuming a quarter-wavelength spacing $d = \lambda_B/4 = \Lambda/2$, we have $\beta d = (\delta + \pi/\Lambda)d = \delta d + \pi/2$. Replacing $e^{j\beta d} = e^{j\delta d + j\pi/2} = j\,e^{j\delta d}$, we obtain:

$$\Gamma_{\text{comp}} = \frac{\Gamma(T^* e^{j\delta d} - T e^{-j\delta d})}{T^* e^{j\delta d} - |\Gamma|^2 T e^{-j\delta d}} \tag{10.5.13}$$

At $\delta = 0$, we have $T = T^* = 1/\cosh|\kappa|l$, and therefore, $\Gamma_{\text{comp}} = 0$. Fig. 10.5.4 depicts the reflectance, $|\Gamma_{\text{comp}}|^2$, and transmittance, $1 - |\Gamma_{\text{comp}}|^2$, for the case $\kappa l = 2$.

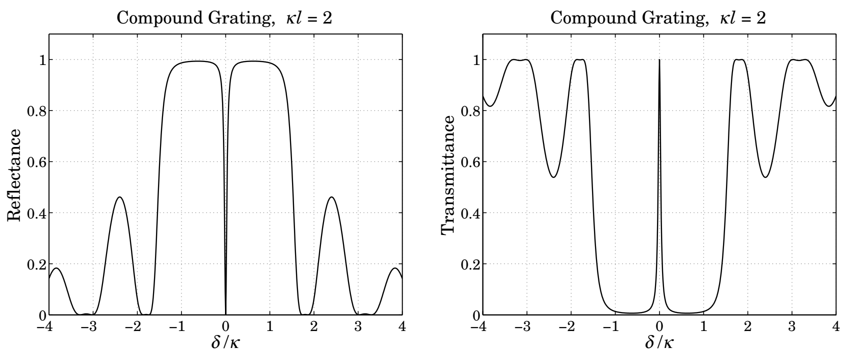


Fig. 10.5.4 Quarter-wave phase-shifted fiber Bragg grating.

Quarter-wave phase-shifted FBGs are similar to the Fabry-Perot resonators discussed in Sec. 5.5. Improved designs having narrow and flat transmission bands can be obtained by cascading several quarter-wave FBGs with different lengths [353–373]. Some applications of FBGs in DWDM systems were pointed out in Sec. 5.7.

10.6 Diffuse Reflection and Transmission

Another example of contra-directional coupling is the two-flux model of Schuster and Kubelka-Munk describing the absorption and multiple scattering of light propagating in a turbid medium [526–542].

The model has a large number of applications, such as radiative transfer in stellar atmospheres, reflectance spectroscopy, reflection and transmission properties of powders, papers, paints, skin tissue, dental materials, and the sea.

The model assumes a simplified parallel-plane geometry, as shown in Fig. 10.6.1. Let $I_\pm(z)$ be the forward and backward radiation intensities per unit frequency interval at location z within the material. The model is described by the two coefficients k, s of absorption and scattering per unit length. For simplicity, we assume that k, s are independent of z.

Within a layer dz, the forward intensity I_+ will be diminished by an amount of $I_+ k\,dz$ due to absorption and an amount of $I_+ s\,dz$ due to scattering, and it will be increased by an amount of $I_- s\,dz$ arising from the backward-moving intensity that is getting scattered

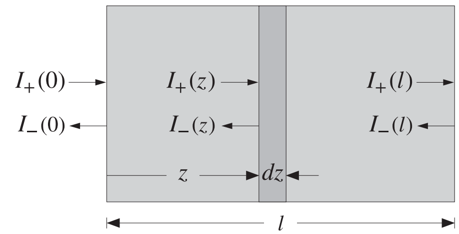


Fig. 10.6.1 Forward and backward intensities in stratified medium.

forward. Similarly, the backward intensity, going from $z + dz$ to z, will be decreased by $I_-(k+s)(-dz)$ and increased by $I_+ s(-dz)$. Thus, the incremental changes are:

$$dI_+ = -(k+s)I_+ dz + sI_- dz$$
$$-dI_- = -(k+s)I_- dz + sI_+ dz$$

or, written in matrix form:

$$\frac{d}{dz}\begin{bmatrix} I_+(z) \\ I_-(z) \end{bmatrix} = -\begin{bmatrix} k+s & -s \\ s & -k-s \end{bmatrix}\begin{bmatrix} I_+(z) \\ I_-(z) \end{bmatrix} \tag{10.6.1}$$

This is similar in structure to Eq. (10.5.3), except the matrix coefficients are real. The solution at distance $z = l$ is obtained in terms of the initial values $I_\pm(0)$ by:

$$\begin{bmatrix} I_+(l) \\ I_-(l) \end{bmatrix} = e^{-Fl}\begin{bmatrix} I_+(0) \\ I_-(0) \end{bmatrix}, \quad \text{with} \quad F = \begin{bmatrix} k+s & -s \\ s & -k-s \end{bmatrix} \tag{10.6.2}$$

The transfer matrix e^{-Fl} is:

$$U = e^{-Fl} = \begin{bmatrix} \cosh\beta l - \dfrac{\alpha}{\beta}\sinh\beta l & \dfrac{s}{\beta}\sinh\beta l \\ -\dfrac{s}{\beta}\sinh\beta l & \cosh\beta l + \dfrac{\alpha}{\beta}\sinh\beta l \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \tag{10.6.3}$$

where $\alpha = k + s$ and $\beta = \sqrt{\alpha^2 - s^2} = \sqrt{k(k+2s)}$.† The transfer matrix is unimodular, that is, $\det U = U_{11}U_{22} - U_{12}U_{21} = 1$.

Of interest are the input reflectance (the albedo) $R = I_-(0)/I_+(0)$ of the length-l structure and its transmittance $T = I_+(l)/I_+(0)$, both expressed in terms of the output, or background, reflectance $R_g = I_-(l)/I_+(l)$. Using Eq. (10.6.2), we find:

$$\begin{aligned} R &= \frac{-U_{21} + U_{11}R_g}{U_{22} - U_{12}R_g} = \frac{s\sinh\beta l + (\beta\cosh\beta l - \alpha\sinh\beta l)R_g}{\beta\cosh\beta l + (\alpha - sR_g)\sinh\beta l} \\ T &= \frac{1}{U_{22} - U_{12}R_g} = \frac{\beta}{\beta\cosh\beta l + (\alpha - sR_g)\sinh\beta l} \end{aligned} \tag{10.6.4}$$

†These are related to the normalized Kubelka [532] variables $a = \alpha/s$, $b = \beta/s$.

The reflectance and transmittance corresponding to a black, non-reflecting, background are obtained by setting $R_g = 0$ in Eq. (10.6.4):

$$\begin{aligned} R_0 &= \frac{-U_{21}}{U_{22}} = \frac{s\sinh\beta l}{\beta\cosh\beta l + \alpha\sinh\beta l} \\ T_0 &= \frac{1}{U_{22}} = \frac{\beta}{\beta\cosh\beta l + \alpha\sinh\beta l} \end{aligned} \tag{10.6.5}$$

The reflectance of an infinitely-thick medium is obtained in the limit $l \to \infty$:

$$R_\infty = \frac{s}{\alpha+\beta} = \frac{s}{k+s+\sqrt{k(k+2s)}} \quad\Rightarrow\quad \frac{k}{s} = \frac{(R_\infty - 1)^2}{2R_\infty} \tag{10.6.6}$$

For the special case of an absorbing but non-scattering medium ($k \neq 0, s = 0$), we have $\alpha = \beta = k$ and the transfer matrix (10.6.3) and Eq. (10.6.4) simplify into:

$$U = e^{-Fl} = \begin{bmatrix} e^{-kl} & 0 \\ 0 & e^{kl} \end{bmatrix}, \quad R = e^{-2kl}R_g\,, \quad T = e^{-kl} \tag{10.6.7}$$

These are in accordance with our expectations for exponential attenuation with distance. The intensities are related by $I_+(l) = e^{-kl}I_+(0)$ and $I_-(l) = e^{kl}I_-(0)$. Thus, the reflectance corresponds to traversing a forward and a reverse path of length l, and the transmittance only a forward path.

Perhaps, the most surprising prediction of this model (first pointed out by Schuster) is that, in the case of a non-absorbing but scattering medium ($k = 0, s \neq 0$), the transmittance is not attenuating exponentially, but rather, inversely with distance. Indeed, setting $\alpha = s$ and taking the limit $\beta^{-1}\sinh\beta l \to l$ as $\beta \to 0$, we find:

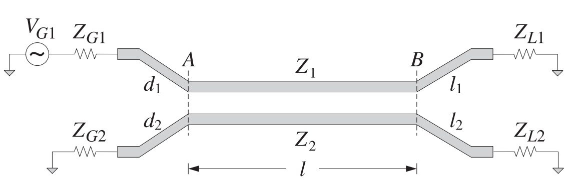
$$U = e^{-Fl} = \begin{bmatrix} 1-sl & sl \\ -sl & 1+sl \end{bmatrix}, \quad R = \frac{sl + (1-sl)R_g}{1+sl-slR_g}\,, \quad T = \frac{1}{1+sl-slR_g} \tag{10.6.8}$$

In particular, for the case of a non-reflecting background, we have:

$$R_0 = \frac{sl}{1+sl}\,, \quad T_0 = \frac{1}{1+sl} \tag{10.6.9}$$

10.7 Problems

10.1 Consider the practical case in which two lines are coupled only over a middle portion of length l, with their beginning and ending segments being uncoupled, as shown below:



Assuming weakly coupled lines, how should Eqs. (10.3.6) and (10.3.9) be modified in this case? [*Hint:* Replace the segments to the left of the reference plane A and to the right of plane B by their Thévenin equivalents.]