4

Reflection and Transmission

4.1 Propagation Matrices

In this chapter, we consider uniform plane waves incident *normally* on material interfaces. Using the boundary conditions for the fields, we will relate the forward-backward fields on one side of the interface to those on the other side, expressing the relationship in terms of a 2×2 *matching matrix*.

If there are several interfaces, we will propagate our forward-backward fields from one interface to the next with the help of a 2×2 *propagation matrix*. The combination of a matching and a propagation matrix relating the fields across different interfaces will be referred to as a *transfer* or transition matrix.

We begin by discussing propagation matrices. Consider an electric field that is linearly polarized in the x-direction and propagating along the z-direction in a lossless (homogeneous and isotropic) dielectric. Setting $\mathbf{E}(z)=\hat{\mathbf{x}}E_x(z)=\hat{\mathbf{x}}E(z)$ and $\mathbf{H}(z)=\hat{\mathbf{y}}H_y(z)=\hat{\mathbf{y}}H(z)$, we have from Eq. (2.2.6):

$$\begin{aligned} E(z) &= E_{0+}e^{-jkz}+E_{0-}e^{jkz} = E_+(z)+E_-(z) \\ H(z) &= \frac{1}{\eta}\big[E_{0+}e^{-jkz}-E_{0-}e^{jkz}\big] = \frac{1}{\eta}\big[E_+(z)-E_-(z)\big] \end{aligned} \tag{4.1.1}$$

where the corresponding forward and backward electric fields at position z are:

$$\begin{aligned} E_+(z) &= E_{0+}e^{-jkz} \\ E_-(z) &= E_{0-}e^{jkz} \end{aligned} \tag{4.1.2}$$

We can also express the fields $E_\pm(z)$ in terms of $E(z),H(z)$. Adding and subtracting the two equations (4.1.1), we find:

$$\begin{aligned} E_+(z) &= \frac{1}{2}\big[E(z)+\eta H(z)\big] \\ E_-(z) &= \frac{1}{2}\big[E(z)-\eta H(z)\big] \end{aligned} \tag{4.1.3}$$

Eqs.(4.1.1) and (4.1.3) can also be written in the convenient matrix forms:

$$\begin{bmatrix} E \\ H \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \eta^{-1} & -\eta^{-1} \end{bmatrix} \begin{bmatrix} E_+ \\ E_- \end{bmatrix}, \quad \begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & \eta \\ 1 & -\eta \end{bmatrix} \begin{bmatrix} E \\ H \end{bmatrix} \tag{4.1.4}$$

Two useful quantities in interface problems are the *wave impedance* at z:

$$\boxed{Z(z)=\frac{E(z)}{H(z)}} \quad \text{(wave impedance)} \tag{4.1.5}$$

and the *reflection coefficient* at position z:

$$\boxed{\Gamma(z)=\frac{E_-(z)}{E_+(z)}} \quad \text{(reflection coefficient)} \tag{4.1.6}$$

Using Eq. (4.1.3), we have:

$$\Gamma = \frac{E_-}{E_+} = \frac{\frac{1}{2}(E-\eta H)}{\frac{1}{2}(E+\eta H)} = \frac{\frac{E}{H}-\eta}{\frac{E}{H}+\eta} = \frac{Z-\eta}{Z+\eta}$$

Similarly, using Eq. (4.1.1) we find:

$$Z = \frac{E}{H} = \frac{E_+ + E_-}{\frac{1}{\eta}(E_+ - E_-)} = \eta\,\frac{1+\frac{E_-}{E_+}}{1-\frac{E_-}{E_+}} = \eta\,\frac{1+\Gamma}{1-\Gamma}$$

Thus, we have the relationships:

$$\boxed{Z(z)=\eta\,\frac{1+\Gamma(z)}{1-\Gamma(z)}} \quad \Leftrightarrow \quad \boxed{\Gamma(z)=\frac{Z(z)-\eta}{Z(z)+\eta}} \tag{4.1.7}$$

Using Eq. (4.1.2), we find:

$$\Gamma(z)=\frac{E_-(z)}{E_+(z)} = \frac{E_{0-}e^{jkz}}{E_{0+}e^{-jkz}} = \Gamma(0)\,e^{2jkz}$$

where $\Gamma(0)=E_{0-}/E_{0+}$ is the reflection coefficient at $z=0$. Thus,

$$\Gamma(z)=\Gamma(0)\,e^{2jkz} \quad \text{(propagation of } \Gamma\text{)} \tag{4.1.8}$$

Applying (4.1.7) at z and $z=0$, we have:

$$\frac{Z(z)-\eta}{Z(z)+\eta} = \Gamma(z) = \Gamma(0)\,e^{2jkz} = \frac{Z(0)-\eta}{Z(0)+\eta}\,e^{2jkz}$$

This may be solved for $Z(z)$ in terms of $Z(0)$, giving after some algebra:

$$Z(z)=\eta\,\frac{Z(0)-j\eta\tan kz}{\eta-jZ(0)\tan kz} \quad \text{(propagation of } Z\text{)} \tag{4.1.9}$$

The reason for introducing so many field quantities is that the three quantities $\{E_+(z), E_-(z), \Gamma(z)\}$ have simple propagation properties, whereas $\{E(z), H(z), Z(z)\}$ do not. On the other hand, $\{E(z), H(z), Z(z)\}$ match simply across interfaces, whereas $\{E_+(z), E_-(z), \Gamma(z)\}$ do not.

Eqs. (4.1.1) and (4.1.2) relate the field quantities at location z to the quantities at $z = 0$. In matching problems, it proves more convenient to be able to relate these quantities at two arbitrary locations.

Fig. 4.1.1 depicts the quantities $\{E(z), H(z), E_+(z), E_-(z), Z(z), \Gamma(z)\}$ at the two locations z_1 and z_2 separated by a distance $l = z_2 - z_1$. Using Eq. (4.1.2), we have for the forward field at these two positions:

$$E_{2+} = E_{0+}e^{-jkz_2}, \qquad E_{1+} = E_{0+}e^{-jkz_1} = E_{0+}e^{-jk(z_2-l)} = e^{jkl}E_{2+}$$

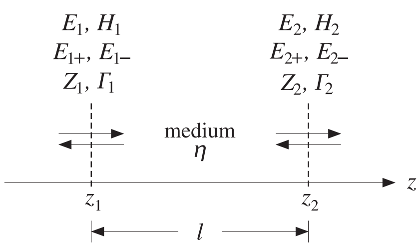


Fig. 4.1.1 Field quantities propagated between two positions in space.

And similarly, $E_{1-} = e^{-jkl}E_{2-}$. Thus,

$$E_{1+} = e^{jkl}E_{2+}, \qquad E_{1-} = e^{-jkl}E_{2-} \tag{4.1.10}$$

and in matrix form:

$$\begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \begin{bmatrix} e^{jkl} & 0 \\ 0 & e^{-jkl} \end{bmatrix} \begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix} \qquad \text{(propagation matrix)} \tag{4.1.11}$$

We will refer to this as the *propagation matrix* for the forward and backward fields. It follows that the reflection coefficients will be related by:

$$\Gamma_1 = \frac{E_{1-}}{E_{1+}} = \frac{E_{2-}e^{-jkl}}{E_{2+}e^{jkl}} = \Gamma_2 e^{-2jkl}, \qquad \text{or,}$$

$$\boxed{\Gamma_1 = \Gamma_2 e^{-2jkl}} \qquad \text{(reflection coefficient propagation)} \tag{4.1.12}$$

Using the matrix relationships (4.1.4) and (4.1.11), we may also express the total electric and magnetic fields E_1, H_1 at position z_1 in terms of E_2, H_2 at position z_2:

$$\begin{bmatrix} E_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \eta^{-1} & -\eta^{-1} \end{bmatrix} \begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \eta^{-1} & -\eta^{-1} \end{bmatrix} \begin{bmatrix} e^{jkl} & 0 \\ 0 & e^{-jkl} \end{bmatrix} \begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ \eta^{-1} & -\eta^{-1} \end{bmatrix} \begin{bmatrix} e^{jkl} & 0 \\ 0 & e^{-jkl} \end{bmatrix} \begin{bmatrix} 1 & \eta \\ 1 & -\eta \end{bmatrix} \begin{bmatrix} E_2 \\ H_2 \end{bmatrix}$$

which gives after some algebra:

$$\begin{bmatrix} E_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} \cos kl & j\eta \sin kl \\ j\eta^{-1}\sin kl & \cos kl \end{bmatrix} \begin{bmatrix} E_2 \\ H_2 \end{bmatrix} \qquad \text{(propagation matrix)} \tag{4.1.13}$$

Writing $\eta = \eta_0/n$, where n is the refractive index of the propagation medium, Eq. (4.1.13) can written in following form, which is useful in analyzing multilayer structures and is common in the thin-film literature [199,201,205,216]:

$$\begin{bmatrix} E_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} \cos\delta & jn^{-1}\eta_0 \sin\delta \\ jn\eta_0^{-1}\sin\delta & \cos\delta \end{bmatrix} \begin{bmatrix} E_2 \\ H_2 \end{bmatrix} \qquad \text{(propagation matrix)} \tag{4.1.14}$$

where δ is the propagation *phase constant*, $\delta = kl = k_0 nl = 2\pi(nl)/\lambda_0$, and nl the *optical length*. Eqs. (4.1.13) and (4.1.5), imply for the propagation of the wave impedance:

$$Z_1 = \frac{E_1}{H_1} = \frac{E_2\cos kl + j\eta H_2 \sin kl}{jE_2\eta^{-1}\sin kl + H_2\cos kl} = \eta \frac{\dfrac{E_2}{H_2}\cos kl + j\eta\sin kl}{\eta\cos kl + j\dfrac{E_2}{H_2}\sin kl}$$

which gives:

$$\boxed{Z_1 = \eta\,\frac{Z_2\cos kl + j\eta\sin kl}{\eta\cos kl + jZ_2\sin kl}} \qquad \text{(impedance propagation)} \tag{4.1.15}$$

It can also be written in the form:

$$\boxed{Z_1 = \eta\,\frac{Z_2 + j\eta\tan kl}{\eta + jZ_2\tan kl}} \qquad \text{(impedance propagation)} \tag{4.1.16}$$

A useful way of expressing Z_1 is in terms of the reflection coefficient Γ_2. Using (4.1.7) and (4.1.12), we have:

$$Z_1 = \eta\,\frac{1+\Gamma_1}{1-\Gamma_1} = \eta\,\frac{1+\Gamma_2 e^{-2jkl}}{1-\Gamma_2 e^{-2jkl}} \qquad \text{or,}$$

$$\boxed{Z_1 = \eta\,\frac{1+\Gamma_2 e^{-2jkl}}{1-\Gamma_2 e^{-2jkl}}} \tag{4.1.17}$$

We mention finally two special propagation cases: the *half-wavelength* and the *quarter-wavelength* cases. When the propagation distance is $l = \lambda/2$, or any integral multiple thereof, the wave impedance and reflection coefficient remain unchanged. Indeed, we have in this case $kl = 2\pi l/\lambda = 2\pi/2 = \pi$ and $2kl = 2\pi$. It follows from Eq. (4.1.12) that $\Gamma_1 = \Gamma_2$ and hence $Z_1 = Z_2$.

If on the other hand $l = \lambda/4$, or any odd integral multiple thereof, then $kl = 2\pi/4 = \pi/2$ and $2kl = \pi$. The reflection coefficient changes sign and the wave impedance inverts:

$$\Gamma_1 = \Gamma_2 e^{-2jkl} = \Gamma_2 e^{-j\pi} = -\Gamma_2 \quad\Rightarrow\quad Z_1 = \eta\,\frac{1+\Gamma_1}{1-\Gamma_1} = \eta\,\frac{1-\Gamma_2}{1+\Gamma_2} = \eta\,\frac{1}{Z_2/\eta} = \frac{\eta^2}{Z_2}$$

Thus, we have in the two cases:

$$\boxed{\begin{aligned} l &= \frac{\lambda}{2} \quad \Rightarrow \quad Z_1 = Z_2, \quad \Gamma_1 = \Gamma_2 \\ l &= \frac{\lambda}{4} \quad \Rightarrow \quad Z_1 = \frac{\eta^2}{Z_2}, \quad \Gamma_1 = -\Gamma_2 \end{aligned}} \tag{4.1.18}$$

4.2 Matching Matrices

Next, we discuss the matching conditions across dielectric interfaces. We consider a planar interface (taken to be the *xy*-plane at some location *z*) separating two dielectric/conducting media with (possibly complex-valued) characteristic impedances η, η', as shown in Fig. 4.2.1.[†]

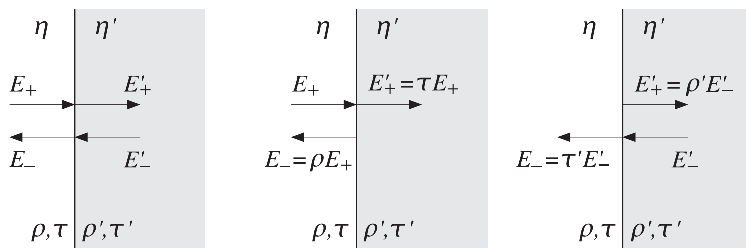


Fig. 4.2.1 Fields across an interface.

Because the normally incident fields are tangential to the interface plane, the boundary conditions require that the *total* electric and magnetic fields be continuous across the two sides of the interface:

$$\boxed{\begin{aligned} E &= E' \\ H &= H' \end{aligned}} \qquad \text{(continuity across interface)} \tag{4.2.1}$$

In terms of the forward and backward electric fields, Eq. (4.2.1) reads:

$$\begin{aligned} E_+ + E_- &= E'_+ + E'_- \\ \frac{1}{\eta}(E_+ - E_-) &= \frac{1}{\eta'}(E'_+ - E'_-) \end{aligned} \tag{4.2.2}$$

Eq. (4.2.2) may be written in a matrix form relating the fields $E_\pm$ on the left of the interface to the fields $E'_\pm$ on the right:

$$\begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \frac{1}{\tau}\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\begin{bmatrix} E'_+ \\ E'_- \end{bmatrix} \qquad \text{(matching matrix)} \tag{4.2.3}$$

and inversely:

[†]The arrows in this figure indicate the directions of propagation, not the direction of the fields—the field vectors are perpendicular to the propagation directions and parallel to the interface plane.

$$\begin{bmatrix} E'_+ \\ E'_- \end{bmatrix} = \frac{1}{\tau'}\begin{bmatrix} 1 & \rho' \\ \rho' & 1 \end{bmatrix}\begin{bmatrix} E_+ \\ E_- \end{bmatrix} \qquad \text{(matching matrix)} \tag{4.2.4}$$

where $\{\rho, \tau\}$ and $\{\rho', \tau'\}$ are the elementary reflection and transmission coefficients from the left and from the right of the interface, defined in terms of η, η' as follows:

$$\boxed{\rho = \frac{\eta' - \eta}{\eta' + \eta}, \quad \tau = \frac{2\eta'}{\eta' + \eta}} \tag{4.2.5}$$

$$\boxed{\rho' = \frac{\eta - \eta'}{\eta + \eta'}, \quad \tau' = \frac{2\eta}{\eta + \eta'}} \tag{4.2.6}$$

Writing $\eta = \eta_0/n$ and $\eta' = \eta_0/n'$, we have in terms of the refractive indices:

$$\boxed{\begin{aligned} \rho &= \frac{n - n'}{n + n'}, \quad \tau = \frac{2n}{n + n'} \\ \rho' &= \frac{n' - n}{n' + n}, \quad \tau' = \frac{2n'}{n' + n} \end{aligned}} \tag{4.2.7}$$

These are also called the Fresnel coefficients. We note various useful relationships:

$$\tau = 1 + \rho, \quad \rho' = -\rho, \quad \tau' = 1 + \rho' = 1 - \rho, \quad \tau\tau' = 1 - \rho^2 \tag{4.2.8}$$

In summary, the total electric and magnetic fields E, H match simply across the interface, whereas the forward/backward fields $E_\pm$ are related by the matching matrices of Eqs. (4.2.3) and (4.2.4). An immediate consequence of Eq. (4.2.1) is that the wave impedance is *continuous* across the interface:

$$Z = \frac{E}{H} = \frac{E'}{H'} = Z'$$

On the other hand, the corresponding reflection coefficients $\Gamma = E_-/E_+$ and $\Gamma' = E'_-/E'_+$ match in a more complicated way. Using Eq. (4.1.7) and the continuity of the wave impedance, we have:

$$\eta\frac{1+\Gamma}{1-\Gamma} = Z = Z' = \eta'\frac{1+\Gamma'}{1-\Gamma'}$$

which can be solved to get:

$$\Gamma = \frac{\rho + \Gamma'}{1 + \rho\Gamma'} \quad \text{and} \quad \Gamma' = \frac{\rho' + \Gamma}{1 + \rho'\Gamma}$$

The same relationship follows also from Eq. (4.2.3):

$$\Gamma = \frac{E_-}{E_+} = \frac{\frac{1}{\tau}(\rho E'_+ + E'_-)}{\frac{1}{\tau}(E'_+ + \rho E'_-)} = \frac{\rho + \frac{E'_-}{E'_+}}{1 + \rho\frac{E'_-}{E'_+}} = \frac{\rho + \Gamma'}{1 + \rho\Gamma'}$$

To summarize, we have the matching conditions for Z and Γ:

$$\boxed{Z = Z'} \quad\Leftrightarrow\quad \boxed{\Gamma = \frac{\rho + \Gamma'}{1 + \rho\Gamma'}} \quad\Leftrightarrow\quad \boxed{\Gamma' = \frac{\rho' + \Gamma}{1 + \rho'\Gamma}} \tag{4.2.9}$$

Two special cases, illustrated in Fig. 4.2.1, are when there is only an incident wave on the interface from the left, so that $E'_- = 0$, and when the incident wave is only from the right, so that $E_+ = 0$. In the first case, we have $\Gamma' = E'_-/E'_+ = 0$, which implies $Z' = \eta'(1+\Gamma')/(1-\Gamma') = \eta'$. The matching conditions give then:

$$Z = Z' = \eta', \quad \Gamma = \frac{\rho + \Gamma'}{1 + \rho\Gamma'} = \rho$$

The matching matrix (4.2.3) implies in this case:

$$\begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \frac{1}{\tau}\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\begin{bmatrix} E'_+ \\ 0 \end{bmatrix} = \frac{1}{\tau}\begin{bmatrix} E'_+ \\ \rho E'_+ \end{bmatrix}$$

Expressing the reflected and transmitted fields E_-, E'_+ in terms of the incident field E_+, we have:

$$\boxed{\begin{aligned} E_- &= \rho E_+ \\ E'_+ &= \tau E_+ \end{aligned}} \qquad \text{(left-incident fields)} \tag{4.2.10}$$

This justifies the terms reflection and transmission coefficients for ρ and τ. In the right-incident case, the condition $E_+ = 0$ implies for Eq. (4.2.4):

$$\begin{bmatrix} E'_+ \\ E'_- \end{bmatrix} = \frac{1}{\tau'}\begin{bmatrix} 1 & \rho' \\ \rho' & 1 \end{bmatrix}\begin{bmatrix} 0 \\ E_- \end{bmatrix} = \frac{1}{\tau'}\begin{bmatrix} \rho' E_- \\ E_- \end{bmatrix}$$

These can be rewritten in the form:

$$\boxed{\begin{aligned} E'_+ &= \rho' E'_- \\ E_- &= \tau' E'_- \end{aligned}} \qquad \text{(right-incident fields)} \tag{4.2.11}$$

which relates the reflected and transmitted fields E'_+, E_- to the incident field E'_-. In this case $\Gamma = E_-/E_+ = \infty$ and the third of Eqs. (4.2.9) gives $\Gamma' = E'_-/E'_+ = 1/\rho'$, which is consistent with Eq. (4.2.11).

When there are incident fields both from both sides, that is, E_+, E'_-, we may invoke the linearity of Maxwell's equations and add the two right-hand sides of Eqs. (4.2.10) and (4.2.11) to obtain the outgoing fields E'_+, E_- in terms of the incident ones:

$$\boxed{\begin{aligned} E'_+ &= \tau E_+ + \rho' E'_- \\ E_- &= \rho E_+ + \tau' E'_- \end{aligned}} \tag{4.2.12}$$

This gives the *scattering matrix* relating the outgoing fields to the incoming ones:

$$\begin{bmatrix} E'_+ \\ E_- \end{bmatrix} = \begin{bmatrix} \tau & \rho' \\ \rho & \tau' \end{bmatrix}\begin{bmatrix} E_+ \\ E'_- \end{bmatrix} \qquad \text{(scattering matrix)} \tag{4.2.13}$$

Using the relationships Eq. (4.2.8), it is easily verified that Eq. (4.2.13) is equivalent to the matching matrix equations (4.2.3) and (4.2.4).

4.3 *Reflected and Transmitted Power*

For waves propagating in the z-direction, the time-averaged Poynting vector has only a z-component:

$$\boldsymbol{\mathcal{P}} = \frac{1}{2}\,\mathrm{Re}\big(\hat{\mathbf{x}}E \times \hat{\mathbf{y}}H^*\big) = \hat{\mathbf{z}}\,\frac{1}{2}\,\mathrm{Re}(EH^*)$$

A direct consequence of the continuity equations (4.2.1) is that the Poynting vector is conserved across the interface. Indeed, we have:

$$\mathcal{P} = \frac{1}{2}\,\mathrm{Re}(EH^*) = \frac{1}{2}\,\mathrm{Re}(E'H'^*) = \mathcal{P}' \tag{4.3.1}$$

In particular, consider the case of a wave incident from a lossless dielectric η onto a lossy dielectric η'. Then, the conservation equation (4.3.1) reads in terms of the forward and backward fields (assuming $E'_- = 0$):

$$\mathcal{P} = \frac{1}{2\eta}\big(|E_+|^2 - |E_-|^2\big) = \mathrm{Re}\Big(\frac{1}{2\eta'}\Big)|E'_+|^2 = \mathcal{P}'$$

The left hand-side is the difference of the incident and the reflected power and represents the amount of power transmitted into the lossy dielectric per unit area. We saw in Sec. 2.6 that this power is completely dissipated into heat inside the lossy dielectric (assuming it is infinite to the right.) Using Eqs. (4.2.10), we find:

$$\mathcal{P} = \frac{1}{2\eta}|E_+|^2(1 - |\rho|^2) = \mathrm{Re}\Big(\frac{1}{2\eta'}\Big)|E_+|^2|\tau|^2 \tag{4.3.2}$$

This equality requires that:

$$\frac{1}{\eta}(1 - |\rho|^2) = \mathrm{Re}\Big(\frac{1}{\eta'}\Big)|\tau|^2 \tag{4.3.3}$$

This can be proved using the definitions (4.2.5). Indeed, we have:

$$\frac{\eta}{\eta'} = \frac{1-\rho}{1+\rho} \quad\Rightarrow\quad \mathrm{Re}\left(\frac{\eta}{\eta'}\right) = \frac{1 - |\rho|^2}{|1+\rho|^2} = \frac{1 - |\rho|^2}{|\tau|^2}$$

which is equivalent to Eq. (4.3.3), if η is lossless (i.e., real.) Defining the incident, reflected, and transmitted powers by

$$\mathcal{P}_{\mathrm{in}} = \frac{1}{2\eta}|E_+|^2$$

$$\mathcal{P}_{\mathrm{ref}} = \frac{1}{2\eta}|E_-|^2 = \frac{1}{2\eta}|E_+|^2|\rho|^2 = \mathcal{P}_{\mathrm{in}}|\rho|^2$$

$$\mathcal{P}_{\mathrm{tr}} = \mathrm{Re}\Big(\frac{1}{2\eta'}\Big)|E'_+|^2 = \mathrm{Re}\Big(\frac{1}{2\eta'}\Big)|E_+|^2|\tau|^2 = \mathcal{P}_{\mathrm{in}}\,\mathrm{Re}\Big(\frac{\eta}{\eta'}\Big)|\tau|^2$$

Then, Eq. (4.3.2) reads $\mathcal{P}_{\mathrm{tr}} = \mathcal{P}_{\mathrm{in}} - \mathcal{P}_{\mathrm{ref}}$. The *power* reflection and transmission coefficients, also known as the *reflectance* and *transmittance*, give the percentage of the incident power that gets reflected and transmitted:

$$\frac{\mathcal{P}_{\rm ref}}{\mathcal{P}_{\rm in}} = |\rho|^2, \qquad \frac{\mathcal{P}_{\rm tr}}{\mathcal{P}_{\rm in}} = 1-|\rho|^2 = \text{Re}\Big(\frac{\eta}{\eta'}\Big)|\tau|^2 = \text{Re}\Big(\frac{n'}{n}\Big)|\tau|^2 \tag{4.3.4}$$

If both dielectrics are lossless, then ρ,τ are real-valued. In this case, if there are incident waves from both sides of the interface, it is straightforward to show that the net power moving towards the z-direction is the same at either side of the interface:

$$\mathcal{P} = \frac{1}{2\eta}\big(|E_+|^2 - |E_-|^2\big) = \frac{1}{2\eta'}\big(|E'_+|^2 - |E'_-|^2\big) = \mathcal{P}' \tag{4.3.5}$$

This follows from the matrix identity satisfied by the matching matrix of Eq. (4.2.3):

$$\frac{1}{\tau^2}\begin{bmatrix} 1 & \rho \\ \rho & 1\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}\begin{bmatrix} 1 & \rho \\ \rho & 1\end{bmatrix} = \frac{\eta}{\eta'}\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix} \tag{4.3.6}$$

If ρ,τ are real, then we have with the help of this identity and Eq. (4.2.3):

$$\begin{aligned}
\mathcal{P} &= \frac{1}{2\eta}\big(|E_+|^2 - |E_-|^2\big) = \frac{1}{2\eta}[E_+^*, E_-^*]\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}\begin{bmatrix} E_+ \\ E_-\end{bmatrix} \\
&= \frac{1}{2\eta}[E'^*_+, E'^*_-]\frac{1}{\tau\tau^*}\begin{bmatrix} 1 & \rho^* \\ \rho^* & 1\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}\begin{bmatrix} 1 & \rho \\ \rho & 1\end{bmatrix}\begin{bmatrix} E'_+ \\ E'_-\end{bmatrix} \\
&= \frac{1}{2\eta}\frac{\eta}{\eta'}[E'^*_+, E'^*_-]\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}\begin{bmatrix} E'_+ \\ E'_-\end{bmatrix} = \frac{1}{2\eta'}\big(|E'_+|^2 - |E'_-|^2\big) = \mathcal{P}'
\end{aligned}$$

Example 4.3.1: Glasses have a refractive index of the order of $n = 1.5$ and dielectric constant $\epsilon = n^2\epsilon_0 = 2.25\epsilon_0$. Calculate the percentages of reflected and transmitted powers for visible light incident on a planar glass interface from air.

Solution: The characteristic impedance of glass will be $\eta = \eta_0/n$. Therefore, the reflection and transmission coefficients can be expressed directly in terms of n, as follows:

$$\rho = \frac{\eta-\eta_0}{\eta+\eta_0} = \frac{n^{-1}-1}{n^{-1}+1} = \frac{1-n}{1+n}, \qquad \tau = 1+\rho = \frac{2}{1+n}$$

For $n = 1.5$, we find $\rho = -0.2$ and $\tau = 0.8$. It follows that the power reflection and transmission coefficients will be

$$|\rho|^2 = 0.04, \qquad 1-|\rho|^2 = 0.96$$

That is, 4% of the incident power is reflected and 96% transmitted. □

Example 4.3.2: A uniform plane wave of frequency f is normally incident from air onto a thick conducting sheet with conductivity σ, and $\epsilon = \epsilon_0$, $\mu = \mu_0$. Show that the proportion of power transmitted into the conductor (and then dissipated into heat) is given approximately by

$$\frac{\mathcal{P}_{\rm tr}}{\mathcal{P}_{\rm in}} = \frac{4R_s}{\eta_0} = \sqrt{\frac{8\omega\epsilon_0}{\sigma}}$$

Calculate this quantity for $f = 1$ GHz and copper $\sigma = 5.8\times 10^7$ Siemens/m.

Solution: For a good conductor, we have $\sqrt{\omega\epsilon_0/\sigma} \ll 1$. It follows from Eq. (2.8.4) that $R_s/\eta_0 = \sqrt{\omega\epsilon_0/2\sigma} \ll 1$. From Eq. (2.8.2), the conductor's characteristic impedance is $\eta_c = R_s(1+j)$. Thus, the quantity $\eta_c/\eta_0 = (1+j)R_s/\eta_0$ is also small. The reflection and transmission coefficients ρ,τ can be expressed to first-order in the quantity η_c/η_0 as follows:

$$\tau = \frac{2\eta_c}{\eta_c+\eta_0} \simeq \frac{2\eta_c}{\eta_0}, \qquad \rho = \tau - 1 \simeq -1 + \frac{2\eta_c}{\eta_0}$$

Similarly, the power transmission coefficient can be approximated as

$$1-|\rho|^2 = 1-|\tau-1|^2 = 1-1-|\tau|^2 + 2\,\text{Re}(\tau) \simeq 2\,\text{Re}(\tau) = 2\frac{2\,\text{Re}(\eta_c)}{\eta_0} = \frac{4R_s}{\eta_0}$$

where we neglected $|\tau|^2$ as it is second order in η_c/η_0. For copper at 1 GHz, we have $\sqrt{\omega\epsilon_0/2\sigma} = 2.19\times 10^{-5}$, which gives $R_s = \eta_0\sqrt{\omega\epsilon_0/2\sigma} = 377\times 2.19\times 10^{-5} = 0.0082\ \Omega$. It follows that $1-|\rho|^2 = 4R_2/\eta_0 = 8.76\times 10^{-5}$.

This represents only a small power loss of 8.76×10^{-3} percent and the sheet acts as very good mirror at microwave frequencies.

On the other hand, at optical frequencies, e.g., $f = 600$ THz corresponding to green light with $\lambda = 500$ nm, the exact equations (2.6.5) yield the value for the characteristic impedance of the sheet $\eta_c = 6.3924 + 6.3888i\ \Omega$ and the reflection coefficient $\rho = -0.9661 + 0.0328i$. The corresponding power loss is $1-|\rho|^2 = 0.065$, or 6.5 percent. Thus, metallic mirrors are fairly lossy at optical frequencies. □

Example 4.3.3: A uniform plane wave of frequency f is normally incident from air onto a thick conductor with conductivity σ, and $\epsilon = \epsilon_0$, $\mu = \mu_0$. Determine the reflected and transmitted electric and magnetic fields to first-order in η_c/η_0 and in the limit of a perfect conductor ($\eta_c = 0$).

Solution: Using the approximations for ρ and τ of the previous example and Eq. (4.2.10), we have for the reflected, transmitted, and total electric fields at the interface:

$$\begin{aligned}
E_- &= \rho E_+ = \Big(-1 + \frac{2\eta_c}{\eta_0}\Big)E_+ \\
E'_+ &= \tau E_+ = \frac{2\eta_c}{\eta_0}E_+ \\
E &= E_+ + E_- = \frac{2\eta_c}{\eta_0}E_+ = E'_+ = E'
\end{aligned}$$

For a perfect conductor, we have $\sigma\to\infty$ and $\eta_c/\eta_0 \to 0$. The corresponding total tangential electric field becomes zero $E = E' = 0$, and $\rho = -1$, $\tau = 0$. For the magnetic fields, we need to develop similar first-order approximations. The incident magnetic field intensity is $H_+ = E_+/\eta_0$. The reflected field becomes to first order:

$$H_- = -\frac{1}{\eta_0}E_- = -\frac{1}{\eta_0}\rho E_+ = -\rho H_+ = \Big(1 - \frac{2\eta_c}{\eta_0}\Big)H_+$$

Similarly, the transmitted field is

$$H'_+ = \frac{1}{\eta_c}E'_+ = \frac{1}{\eta_c}\tau E_+ = \frac{\eta_0}{\eta_c}\tau H_+ = \frac{\eta_0}{\eta_c}\frac{2\eta_c}{\eta_c+\eta_0}H_+ = \frac{2\eta_0}{\eta_c+\eta_0}H_+ \simeq 2\left(1-\frac{\eta_c}{\eta_0}\right)H_+$$

The total tangential field at the interface will be:

$$H = H_+ + H_- = 2\left(1-\frac{\eta_c}{\eta_0}\right)H_+ = H'_+ = H'$$

In the perfect conductor limit, we find $H = H' = 2H_+$. As we saw in Sec. 2.6, the fields just inside the conductor, E'_+, H'_+, will attenuate while they propagate. Assuming the interface is at $z = 0$, we have:

$$E'_+(z) = E'_+ e^{-\alpha z}e^{-j\beta z}, \qquad H'_+(z) = H'_+ e^{-\alpha z}e^{-j\beta z}$$

where $\alpha = \beta = (1-j)/\delta$, and δ is the skin depth $\delta = \sqrt{\omega\mu\sigma/2}$. We saw in Sec. 2.6 that the effective surface current is equal in magnitude to the magnetic field at $z = 0$, that is, $J_s = H'_+$. Because of the boundary condition $H = H' = H'_+$, we obtain the result $J_s = H$, or vectorially, $\mathbf{J}_s = \mathbf{H}\times\hat{\mathbf{z}} = \hat{\mathbf{n}}\times\mathbf{H}$, where $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$ is the outward normal to the conductor.

This result provides a justification of the boundary condition $\mathbf{J}_s = \hat{\mathbf{n}}\times\mathbf{H}$ at an interface with a perfect conductor. □

4.4 *Single Dielectric Slab*

Multiple interface problems can be handled in a straightforward way with the help of the matching and propagation matrices. For example, Fig. 4.4.1 shows a two-interface problem with a dielectric slab η_1 separating the semi-infinite media η_a and η_b.

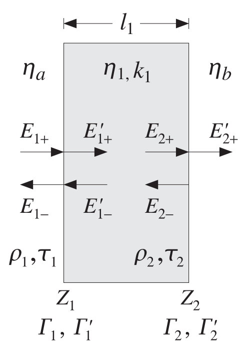


Fig. 4.4.1 Single dielectric slab.

Let l_1 be the width of the slab, $k_1 = \omega/c_1$ the propagation wavenumber, and $\lambda_1 = 2\pi/k_1$ the corresponding wavelength within the slab. We have $\lambda_1 = \lambda_0/n_1$, where λ_0 is the free-space wavelength and n_1 the refractive index of the slab. We assume the incident field is from the left medium η_a, and thus, in medium η_b there is only a forward wave.

Let ρ_1, ρ_2 be the elementary reflection coefficients from the left sides of the two interfaces, and let τ_1, τ_2 be the corresponding transmission coefficients:

$$\rho_1 = \frac{\eta_1-\eta_a}{\eta_1+\eta_a}, \quad \rho_2 = \frac{\eta_b-\eta_1}{\eta_b+\eta_1}, \quad \tau_1 = 1+\rho_1, \quad \tau_2 = 1+\rho_2 \tag{4.4.1}$$

To determine the reflection coefficient Γ_1 into medium η_a, we apply Eq. (4.2.9) to relate Γ_1 to the reflection coefficient Γ'_1 at the right-side of the first interface. Then, we propagate to the left of the second interface with Eq. (4.1.12) to get:

$$\Gamma_1 = \frac{\rho_1+\Gamma'_1}{1+\rho_1\Gamma'_1} = \frac{\rho_1+\Gamma_2 e^{-2jk_1l_1}}{1+\rho_1\Gamma_2 e^{-2jk_1l_1}} \tag{4.4.2}$$

At the second interface, we apply Eq. (4.2.9) again to relate Γ_2 to Γ'_2. Because there are no backward-moving waves in medium η_b, we have $\Gamma'_2 = 0$. Thus,

$$\Gamma_2 = \frac{\rho_2+\Gamma'_2}{1+\rho_2\Gamma'_2} = \rho_2$$

We finally find for Γ_1:

$$\boxed{\Gamma_1 = \frac{\rho_1+\rho_2 e^{-2jk_1l_1}}{1+\rho_1\rho_2 e^{-2jk_1l_1}}} \tag{4.4.3}$$

This expression can be thought of as function of frequency. Assuming a lossless medium η_1, we have $2k_1l_1 = \omega(2l_1/c_1) = \omega T$, where $T = 2l_1/c_1 = 2(n_1l_1)/c_0$ is the *two-way travel time delay* through medium η_1. Thus, we can write:

$$\boxed{\Gamma_1(\omega) = \frac{\rho_1+\rho_2 e^{-j\omega T}}{1+\rho_1\rho_2 e^{-j\omega T}}} \tag{4.4.4}$$

This can also be expressed as a z-transform. Denoting the two-way travel time delay in the z-domain by $z^{-1} = e^{-j\omega T} = e^{-2jk_1l_1}$, we may rewrite Eq. (4.4.4) as the first-order digital filter *transfer function*:

$$\boxed{\Gamma_1(z) = \frac{\rho_1+\rho_2 z^{-1}}{1+\rho_1\rho_2 z^{-1}}} \tag{4.4.5}$$

An alternative way to derive Eq. (4.4.3) is working with wave impedances, which are continuous across interfaces. The wave impedance at interface-2 is $Z_2 = Z'_2$, but $Z'_2 = \eta_b$ because there is no backward wave in medium η_b. Thus, $Z_2 = \eta_b$. Using the propagation equation for impedances, we find:

$$Z_1 = Z'_1 = \eta_1\frac{Z_2+j\eta_1\tan k_1l_1}{\eta_1+jZ_2\tan k_1l_1} = \eta_1\frac{\eta_b+j\eta_1\tan k_1l_1}{\eta_1+j\eta_b\tan k_1l_1}$$

Inserting this into $\Gamma_1 = (Z_1-\eta_a)/(Z_1+\eta_a)$ gives Eq. (4.4.3). Working with wave impedances is always more convenient if the interfaces are positioned at half- or quarter-wavelength spacings.

If we wish to determine the overall *transmission response* into medium η_b, that is, the quantity $\mathcal{T} = E'_{2+}/E_{1+}$, then we must work with the matrix formulation. Starting at the left interface and successively applying the matching and propagation matrices, we obtain:

$$\begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \frac{1}{\tau_1}\begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix}\begin{bmatrix} E'_{1+} \\ E'_{1-} \end{bmatrix} = \frac{1}{\tau_1}\begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix}\begin{bmatrix} e^{jk_1l_1} & 0 \\ 0 & e^{-jk_1l_1} \end{bmatrix}\begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix}$$
$$= \frac{1}{\tau_1}\begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix}\begin{bmatrix} e^{jk_1l_1} & 0 \\ 0 & e^{-jk_1l_1} \end{bmatrix}\frac{1}{\tau_2}\begin{bmatrix} 1 & \rho_2 \\ \rho_2 & 1 \end{bmatrix}\begin{bmatrix} E'_{2+} \\ 0 \end{bmatrix}$$

where we set $E'_{2-} = 0$ by assumption. Multiplying the matrix factors out, we obtain:

$$E_{1+} = \frac{e^{jk_1l_1}}{\tau_1\tau_2}\big(1 + \rho_1\rho_2 e^{-2jk_1l_1}\big)E'_{2+}$$
$$E_{1-} = \frac{e^{jk_1l_1}}{\tau_1\tau_2}\big(\rho_1 + \rho_2 e^{-2jk_1l_1}\big)E'_{2+}$$

These may be solved for the reflection and transmission responses:

$$\Gamma_1 = \frac{E_{1-}}{E_{1+}} = \frac{\rho_1 + \rho_2 e^{-2jk_1l_1}}{1 + \rho_1\rho_2 e^{-2jk_1l_1}}$$
$$\mathcal{T} = \frac{E'_{2+}}{E_{1+}} = \frac{\tau_1\tau_2 e^{-jk_1l_1}}{1 + \rho_1\rho_2 e^{-2jk_1l_1}} \qquad (4.4.6)$$

The transmission response has an overall delay factor of $e^{-jk_1l_1} = e^{-j\omega T/2}$, representing the *one-way* travel time delay through medium η_1.

For convenience, we summarize the match-and-propagate equations relating the field quantities at the *left* of interface-1 to those at the *left* of interface-2. The forward and backward electric fields are related by the *transfer matrix*:

$$\begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \frac{1}{\tau_1}\begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix}\begin{bmatrix} e^{jk_1l_1} & 0 \\ 0 & e^{-jk_1l_1} \end{bmatrix}\begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix}$$
$$\begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \frac{1}{\tau_1}\begin{bmatrix} e^{jk_1l_1} & \rho_1 e^{-jk_1l_1} \\ \rho_1 e^{jk_1l_1} & e^{-jk_1l_1} \end{bmatrix}\begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix} \qquad (4.4.7)$$

The reflection responses are related by Eq. (4.4.2):

$$\Gamma_1 = \frac{\rho_1 + \Gamma_2 e^{-2jk_1l_1}}{1 + \rho_1\Gamma_2 e^{-2jk_1l_1}} \qquad (4.4.8)$$

The total electric and magnetic fields at the two interfaces are continuous across the interfaces and are related by Eq. (4.1.13):

$$\begin{bmatrix} E_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} \cos k_1l_1 & j\eta_1\sin k_1l_1 \\ j\eta_1^{-1}\sin k_1l_1 & \cos k_1l_1 \end{bmatrix}\begin{bmatrix} E_2 \\ H_2 \end{bmatrix} \qquad (4.4.9)$$

Eqs. (4.4.7)–(4.4.9) are valid in general, regardless of what is to the right of the second interface. There could be a semi-infinite uniform medium or any combination of multiple slabs. These equations were simplified in the single-slab case because we assumed that there was a uniform medium to the right and that there were no backward-moving waves.

For lossless media, energy conservation states that the energy flux into medium η_1 must equal the energy flux out of it. It is equivalent to the following relationship between Γ and $\mathcal{T}$, which can proved using Eq. (4.4.6):

$$\frac{1}{\eta_a}\big(1 - |\Gamma_1|^2\big) = \frac{1}{\eta_b}|\mathcal{T}|^2 \qquad (4.4.10)$$

Thus, if we call $|\Gamma_1|^2$ the *reflectance* of the slab, representing the fraction of the incident power that gets reflected back into medium η_a, then the quantity

$$\boxed{1 - |\Gamma_1|^2 = \frac{\eta_a}{\eta_b}|\mathcal{T}|^2 = \frac{n_b}{n_a}|\mathcal{T}|^2} \qquad (4.4.11)$$

will be the *transmittance* of the slab, representing the fraction of the incident power that gets transmitted through into the right medium η_b. The presence of the factors η_a, η_b can be can be understood as follows:

$$\frac{\mathcal{P}_{\text{transmitted}}}{\mathcal{P}_{\text{incident}}} = \frac{\dfrac{1}{2\eta_b}|E'_{2+}|^2}{\dfrac{1}{2\eta_a}|E_{1+}|^2} = \frac{\eta_a}{\eta_b}|\mathcal{T}|^2$$

4.5 Reflectionless Slab

The *zeros* of the transfer function (4.4.5) correspond to a reflectionless interface. Such zeros can be realized exactly only in two special cases, that is, for slabs that have either half-wavelength or quarter-wavelength thickness. It is evident from Eq. (4.4.5) that a zero will occur if $\rho_1 + \rho_2 z^{-1} = 0$, which gives the condition:

$$z = e^{2jk_1l_1} = -\frac{\rho_2}{\rho_1} \qquad (4.5.1)$$

Because the right-hand side is real-valued and the left-hand side has unit magnitude, this condition can be satisfied only in the following two cases:

$$z = e^{2jk_1l_1} = 1, \qquad \rho_2 = -\rho_1, \qquad \text{(half-wavelength thickness)}$$
$$z = e^{2jk_1l_1} = -1, \qquad \rho_2 = \rho_1, \qquad \text{(quarter-wavelength thickness)}$$

The first case requires that $2k_1l_1$ be an integral multiple of 2π, that is, $2k_1l_1 = 2m\pi$, where m is an integer. This gives the *half-wavelength* condition $l_1 = m\lambda_1/2$, where λ_1 is the wavelength in medium-1. In addition, the condition $\rho_2 = -\rho_1$ requires that:

$$\frac{\eta_b - \eta_1}{\eta_b + \eta_1} = \rho_2 = -\rho_1 = \frac{\eta_a - \eta_1}{\eta_a + \eta_1} \quad \Leftrightarrow \quad \eta_a = \eta_b$$

that is, the media to the left and right of the slab must be the *same*. The second possibility requires $e^{2jk_1l_1} = -1$, or that $2k_1l_1$ be an odd multiple of π, that is, $2k_1l_1 = (2m+1)\pi$, which translates into the *quarter-wavelength* condition $l_1 = (2m+1)\lambda_1/4$. Furthermore, the condition $\rho_2 = \rho_1$ requires:

$$\frac{\eta_b - \eta_1}{\eta_b + \eta_1} = \rho_2 = \rho_1 = \frac{\eta_1 - \eta_a}{\eta_1 + \eta_a} \quad \Leftrightarrow \quad \eta_1^2 = \eta_a\eta_b$$

To summarize, a reflectionless slab, $\Gamma_1 = 0$, can be realized only in the two cases:

$$\begin{aligned} &\text{half-wave:} && l_1 = m\,\frac{\lambda_1}{2}\,, && \eta_1 \text{ arbitrary}, && \eta_a = \eta_b \\ &\text{quarter-wave:} && l_1 = (2m+1)\,\frac{\lambda_1}{4}\,, && \eta_1 = \sqrt{\eta_a\eta_b}\,, && \eta_a, \eta_b \text{ arbitrary} \end{aligned} \tag{4.5.2}$$

An equivalent way of stating these conditions is to say that the *optical length* of the slab must be a half or quarter of the *free-space* wavelength λ_0. Indeed, if n_1 is the refractive index of the slab, then its optical length is $n_1 l_1$, and in the half-wavelength case we have $n_1 l_1 = n_1 m\lambda_1/2 = m\lambda_0/2$, where we used $\lambda_1 = \lambda_0/n_1$. Similarly, we have $n_1 l_1 = (2m+1)\lambda_0/4$ in the quarter-wavelength case. In terms of the refractive indices, Eq. (4.5.2) reads:

$$\boxed{\begin{aligned} &\text{half-wave:} && n_1 l_1 = m\,\frac{\lambda_0}{2}\,, && n_1 \text{ arbitrary}, && n_a = n_b \\ &\text{quarter-wave:} && n_1 l_1 = (2m+1)\,\frac{\lambda_0}{4}\,, && n_1 = \sqrt{n_a n_b}\,, && n_a, n_b \text{ arbitrary} \end{aligned}} \tag{4.5.3}$$

The reflectionless matching condition can also be derived by working with wave impedances. For half-wavelength spacing, we have from Eq. (4.1.18) $Z_1 = Z_2 = \eta_b$. The condition $\Gamma_1 = 0$ requires $Z_1 = \eta_a$, thus, matching occurs if $\eta_a = \eta_b$. Similarly, for the quarter-wavelength case, we have $Z_1 = \eta_1^2/Z_2 = \eta_1^2/\eta_b = \eta_a$.

We emphasize that the reflectionless response $\Gamma_1 = 0$ is obtained only at certain slab widths (half- or quarter-wavelength), or equivalently, at certain operating frequencies. These operating frequencies correspond to $\omega T = 2m\pi$, or, $\omega T = (2m+1)\pi$, that is, $\omega = 2m\pi/T = m\omega_0$, or, $\omega = (2m+1)\omega_0/2$, where we defined $\omega_0 = 2\pi/T$.

The dependence on l_1 or ω can be seen from Eq. (4.4.5). For the half-wavelength case, we substitute $\rho_2 = -\rho_1$ and for the quarter-wavelength case, $\rho_2 = \rho_1$. Then, the reflection transfer functions become:

$$\begin{aligned} \Gamma_1(z) &= \frac{\rho_1(1 - z^{-1})}{1 - \rho_1^2 z^{-1}}\,, \quad \text{(half-wave)} \\ \Gamma_1(z) &= \frac{\rho_1(1 + z^{-1})}{1 + \rho_1^2 z^{-1}}\,, \quad \text{(quarter-wave)} \end{aligned} \tag{4.5.4}$$

where $z = e^{2jk_1 l_1} = e^{j\omega T}$. The magnitude-square responses then take the form:

$$\begin{aligned} |\Gamma_1|^2 &= \frac{2\rho_1^2\big(1 - \cos(2k_1 l_1)\big)}{1 - 2\rho_1^2\cos(2k_1 l_1) + \rho_1^4} = \frac{2\rho_1^2(1 - \cos\omega T)}{1 - 2\rho_1^2\cos\omega T + \rho_1^4}\,, \quad \text{(half-wave)} \\ |\Gamma_1|^2 &= \frac{2\rho_1^2\big(1 + \cos(2k_1 l_1)\big)}{1 + 2\rho_1^2\cos(2k_1 l_1) + \rho_1^4} = \frac{2\rho_1^2(1 + \cos\omega T)}{1 + 2\rho_1^2\cos\omega T + \rho_1^4}\,, \quad \text{(quarter-wave)} \end{aligned} \tag{4.5.5}$$

These expressions are periodic in l_1 with period $\lambda_1/2$, and periodic in ω with period $\omega_0 = 2\pi/T$. In DSP language, the slab acts as a digital filter with sampling frequency ω_0. The maximum reflectivity occurs at $z = -1$ and $z = 1$ for the half- and quarter-wavelength cases. The maximum squared responses are in either case:

$$|\Gamma_1|^2_{\max} = \frac{4\rho_1^2}{(1 + \rho_1^2)^2}$$

Fig. 4.5.1 shows the magnitude responses for the three values of the reflection coefficient: $|\rho_1| = 0.9$, 0.7, and 0.5. The closer ρ_1 is to unity, the narrower are the reflectionless notches.

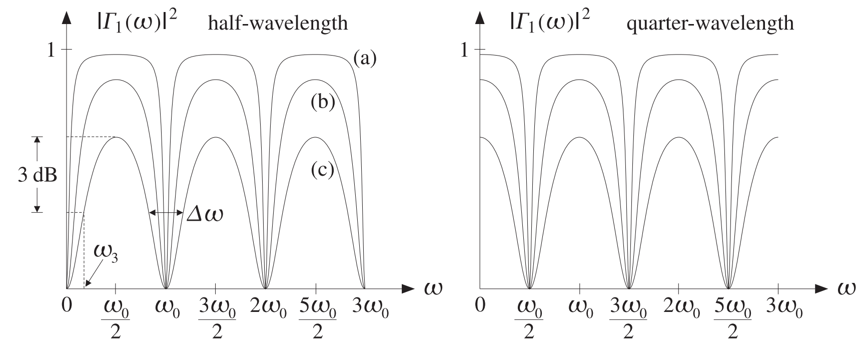


Fig. 4.5.1 Reflection responses $|\Gamma(\omega)|^2$. (a) $|\rho_1| = 0.9$, (b) $|\rho_1| = 0.7$, (c) $|\rho_1| = 0.5$.

It is evident from these figures that for the same value of ρ_1, the half- and quarter-wavelength cases have the same notch widths. A standard measure for the width is the 3-dB width, which for the half-wavelength case is twice the 3-dB frequency ω_3, that is, $\Delta\omega = 2\omega_3$, as shown in Fig. 4.5.1 for the case $|\rho_1| = 0.5$. The frequency ω_3 is determined by the 3-dB half-power condition:

$$|\Gamma_1(\omega_3)|^2 = \frac{1}{2}\,|\Gamma_1|^2_{\max}$$

or, equivalently:

$$\frac{2\rho_1^2(1 - \cos\omega_3 T)}{1 - 2\rho_1^2\cos\omega_3 T + \rho_1^4} = \frac{1}{2}\,\frac{4\rho_1^2}{(1 + \rho_1^2)^2}$$

Solving for the quantity $\cos\omega_3 T = \cos(\Delta\omega T/2)$, we find:

$$\cos\Big(\frac{\Delta\omega T}{2}\Big) = \frac{2\rho_1^2}{1 + \rho_1^4} \quad \Leftrightarrow \quad \tan\Big(\frac{\Delta\omega T}{4}\Big) = \frac{1 - \rho_1^2}{1 + \rho_1^2} \tag{4.5.6}$$

If ρ_1^2 is very near unity, then $1 - \rho_1^2$ and $\Delta\omega$ become small, and we may use the approximation $\tan x \simeq x$ to get:

$$\frac{\Delta\omega T}{4} \simeq \frac{1 - \rho_1^2}{1 + \rho_1^2} \simeq \frac{1 - \rho_1^2}{2}$$

which gives the approximation:

$$\Delta\omega T = 2(1-\rho_1^2) \tag{4.5.7}$$

This is a standard approximation for digital filters relating the 3-dB width of a pole peak to the radius of the pole [52]. For any desired value of the bandwidth $\Delta\omega$, Eq. (4.5.6) or (4.5.7) may be thought of as a design condition that determines ρ_1.

Fig. 4.5.2 shows the corresponding *transmittances* $1-|\Gamma_1(\omega)|^2$ of the slabs. The transmission response acts as a periodic bandpass filter. This is the simplest example of a so-called *Fabry-Perot interference filter* or *Fabry-Perot resonator*. Such filters find application in the spectroscopic analysis of materials. We discuss them further in Chap. 5.

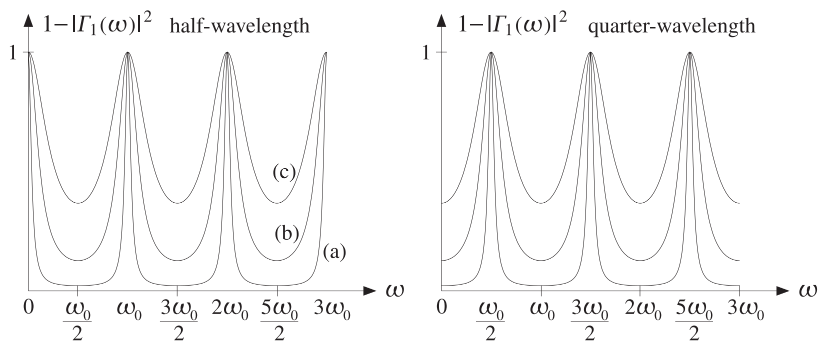


Fig. 4.5.2 Transmittance of half- and quarter-wavelength dielectric slab.

Using Eq. (4.5.5), we may express the frequency response of the half-wavelength transmittance filter in the following equivalent forms:

$$1-|\Gamma_1(\omega)|^2 = \frac{(1-\rho_1^2)^2}{1-2\rho_1^2\cos\omega T+\rho_1^4} = \frac{1}{1+\mathcal{F}\sin^2(\omega T/2)} \tag{4.5.8}$$

where the $\mathcal{F}$ is called the *finesse* in the Fabry-Perot context and is defined by:

$$\mathcal{F} = \frac{2\rho_1^2}{(1-\rho_1^2)^2}$$

The finesse is a measure of the peak width, with larger values of $\mathcal{F}$ corresponding to narrower peaks. The connection of $\mathcal{F}$ to the 3-dB width (4.5.6) is easily found to be:

$$\tan\left(\frac{\Delta\omega T}{4}\right) = \frac{1-\rho_1^2}{1+\rho_1^2} = \frac{1}{\sqrt{2+\mathcal{F}}} \tag{4.5.9}$$

Quarter-wavelength slabs may be used to design anti-reflection coatings for lenses, so that all incident light on a lens gets through. Half-wavelength slabs, which require that the medium be the same on either side of the slab, may be used in designing radar domes (radomes) protecting microwave antennas, so that the radiated signal from the antenna goes through the radome wall without getting reflected back towards the antenna.

Example 4.5.1: Determine the reflection coefficients of half- and quarter-wave slabs that do not necessarily satisfy the impedance conditions of Eq. (4.5.2).

Solution: The reflection response is given in general by Eq. (4.4.6). For the half-wavelength case, we have $e^{2jk_1l_1}=1$ and we obtain:

$$\Gamma_1 = \frac{\rho_1+\rho_2}{1+\rho_1\rho_2} = \frac{\dfrac{\eta_1-\eta_a}{\eta_1+\eta_a}+\dfrac{\eta_b-\eta_1}{\eta_b+\eta_1}}{1+\dfrac{\eta_1-\eta_a}{\eta_1+\eta_a}\dfrac{\eta_b-\eta_1}{\eta_b+\eta_1}} = \frac{\eta_b-\eta_a}{\eta_b+\eta_a} = \frac{n_a-n_b}{n_a+n_b}$$

This is the same as if the slab were absent. For this reason, half-wavelength slabs are sometimes referred to as *absentee* layers. Similarly, in the quarter-wavelength case, we have $e^{2jk_1l_1}=-1$ and find:

$$\Gamma_1 = \frac{\rho_1-\rho_2}{1-\rho_1\rho_2} = \frac{\eta_1^2-\eta_a\eta_b}{\eta_1^2+\eta_a\eta_b} = \frac{n_an_b-n_1^2}{n_an_b+n_1^2}$$

The slab becomes reflectionless if the conditions (4.5.2) are satisfied. □

Example 4.5.2: *Antireflection Coating.* Determine the refractive index of a quarter-wave antireflection coating on a glass substrate with index 1.5.

Solution: From Eq. (4.5.3), we have with $n_a=1$ and $n_b=1.5$:

$$n_1 = \sqrt{n_an_b} = \sqrt{1.5} = 1.22$$

The closest refractive index that can be obtained is that of cryolite (Na_3AlF_6) with $n_1=1.35$ and magnesium fluoride (MgF_2) with $n_1=1.38$. Magnesium fluoride is usually preferred because of its durability. Such a slab will have a reflection coefficient as given by the previous example:

$$\Gamma_1 = \frac{\rho_1-\rho_2}{1-\rho_1\rho_2} = \frac{\eta_1^2-\eta_a\eta_b}{\eta_1^2+\eta_a\eta_b} = \frac{n_an_b-n_1^2}{n_an_b+n_1^2} = \frac{1.5-1.38^2}{1.5+1.38^2} = -0.118$$

with reflectance $|\Gamma|^2 = 0.014$, or 1.4 percent. This is to be compared to the 4 percent reflectance of uncoated glass that we determined in Example 4.3.1.

Fig. 4.5.3 shows the reflectance $|\Gamma(\lambda)|^2$ as a function of the free-space wavelength λ. The reflectance remains less than one or two percent in the two cases, over almost the entire visible spectrum.

The slabs were designed to have quarter-wavelength thickness at $\lambda_0 = 550$ nm, that is, the optical length was $n_1l_1 = \lambda_0/4$, resulting in $l_1 = 112.71$ nm and 99.64 nm in the two cases of $n_1=1.22$ and $n_1=1.38$. Such extremely thin dielectric films are fabricated by means of a thermal evaporation process [199,201].

The MATLAB code used to generate this example was as follows:

```
n = [1, 1.22, 1.50]; L = 1/4;                  refractive indices and optical length
lambda = linspace(400,700,101) / 550;          visible spectrum wavelengths
Gamma1 = multidiel(n, L, lambda);              reflection response of slab
```

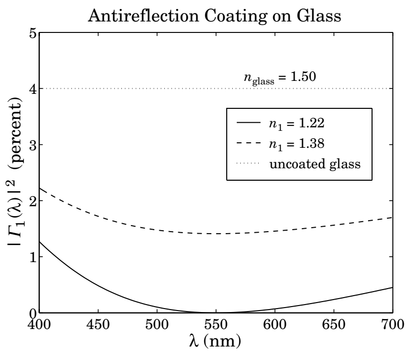


Fig. 4.5.3 Reflectance over the visible spectrum.

The syntax and use of the function `multidiel` is discussed in Sec. 5.1. The dependence of Γ on λ comes through the quantity $k_1 l_1 = 2\pi(n_1 l_1)/\lambda$. Since $n_1 l_1 = \lambda_0/4$, we have $k_1 l_1 = 0.5\pi\lambda_0/\lambda$. □

Example 4.5.3: *Thick Glasses.* Interference phenomena, such as those arising from the multiple reflections within a slab, are not observed if the slabs are "thick" (compared to the wavelength.) For example, typical glass windows seem perfectly transparent.

If one had a glass plate of thickness, say, of $l = 1.5$ mm and index $n = 1.5$, it would have optical length $nl = 1.5\times 1.5 = 2.25$ mm $= 225\times 10^4$ nm. At an operating wavelength of $\lambda_0 = 450$ nm, the glass plate would act as a half-wave transparent slab with $nl = 10^4(\lambda_0/2)$, that is, 10^4 half-wavelengths long.

Such plate would be very difficult to construct as it would require that l be built with an accuracy of a few percent of $\lambda_0/2$. For example, assuming $n(\Delta l) = 0.01(\lambda_0/2)$, the plate should be constructed with an accuracy of one part in a million: $\Delta l/l = n\Delta l/(nl) = 0.01/10^4 = 10^{-6}$. (That is why thin films are constructed by a carefully controlled evaporation process.)

More realistically, a typical glass plate can be constructed with an accuracy of one part in a thousand, $\Delta l/l = 10^{-3}$, which would mean that within the manufacturing uncertainty Δl, there would still be ten half-wavelengths, $n\Delta\lambda = 10^{-3}(nl) = 10(\lambda_0/2)$.

The overall power reflection response will be obtained by *averaging* $|\Gamma_1|^2$ over several $\lambda_0/2$ cycles, such as the above ten. Because of periodicity, the average of $|\Gamma_1|^2$ over several cycles is the same as the average over one cycle, that is,

$$\overline{|\Gamma_1|^2} = \frac{1}{\omega_0}\int_0^{\omega_0} |\Gamma_1(\omega)|^2\, d\omega$$

where $\omega_0 = 2\pi/T$ and T is the two-way travel-time delay. Using either of the two expressions in Eq. (4.5.5), this integral can be done exactly resulting in the average reflectance and transmittance:

$$\overline{|\Gamma_1|^2} = \frac{2\rho_1^2}{1+\rho_1^2}\,, \quad 1 - \overline{|\Gamma_1|^2} = \frac{1-\rho_1^2}{1+\rho_1^2} = \frac{2n}{n^2+1} \tag{4.5.10}$$

where we used $\rho_1 = (1-n)/(1+n)$. This explains why glass windows do not exhibit a frequency-selective behavior as predicted by Eq. (4.5.5). For $n = 1.5$, we find $1 - \overline{|\Gamma_1|^2} = 0.9231$, that is, 92.31% of the incident light is transmitted through the plate.

The same expressions for the average reflectance and transmittance can be obtained by summing incoherently all the multiple reflections within the slab, that is, summing the multiple reflections of power instead of field amplitudes. The timing diagram for such multiple reflections is shown in Fig. 4.6.1.

Indeed, if we denote by $p_r = \rho_1^2$ and $p_t = 1 - p_r = 1 - \rho_1^2$, the power reflection and transmission coefficients, then the first reflection of power will be p_r. The power transmitted through the left interface will be p_t and through the second interface p_t^2 (assuming the same medium to the right.) The reflected power at the second interface will be $p_t p_r$ and will come back and transmit through the left interface giving $p_t^2 p_r$.

Similarly, after a second round trip, the reflected power will be $p_t^2 p_r^3$, while the transmitted power to the right of the second interface will be $p_t^2 p_r^2$, and so on. Summing up all the reflected powers to the left and those transmitted to the right, we find:

$$\overline{|\Gamma_1|^2} = p_r + p_t^2 p_r + p_t^2 p_r^3 + p_t^2 p_r^5 + \cdots = p_r + \frac{p_t^2 p_r}{1-p_r^2} = \frac{2p_r}{1+p_r}$$

$$1 - \overline{|\Gamma_1|^2} = p_t^2 + p_t^2 p_r^2 + p_t^2 p_r^4 + \cdots = \frac{p_t^2}{1-p_r^2} = \frac{1-p_r}{1+p_r}$$

where we used $p_t = 1 - p_r$. These are equivalent to Eqs. (4.5.10). □

Example 4.5.4: *Radomes.* A radome protecting a microwave transmitter has $\epsilon = 4\epsilon_0$ and is designed as a half-wavelength reflectionless slab at the operating frequency of 10 GHz. Determine its thickness.

Next, suppose that the operating frequency is 1% off its nominal value of 10 GHz. Calculate the percentage of reflected power back towards the transmitting antenna.

Determine the operating *bandwidth* as that frequency interval about the 10 GHz operating frequency within which the reflected power remains at least 30 dB below the incident power.

Solution: The free-space wavelength is $\lambda_0 = c_0/f_0 = 30$ GHz cm/10 GHz = 3 cm. The refractive index of the slab is $n = 2$ and the wavelength inside it, $\lambda_1 = \lambda_0/n = 3/2 = 1.5$ cm. Thus, the slab thickness will be the half-wavelength $l_1 = \lambda_1/2 = 0.75$ cm, or any other integral multiple of this.

Assume now that the operating frequency is $\omega = \omega_0 + \delta\omega$, where $\omega_0 = 2\pi f_0 = 2\pi/T$. Denoting $\delta = \delta\omega/\omega_0$, we can write $\omega = \omega_0(1+\delta)$. The numerical value of δ is very small, $\delta = 1\% = 0.01$. Therefore, we can do a first-order calculation in δ. The reflection coefficient ρ_1 and reflection response Γ are:

$$\rho_1 = \frac{\eta - \eta_0}{\eta + \eta_0} = \frac{0.5 - 1}{0.5 + 1} = -\frac{1}{3}\,, \qquad \Gamma_1(\omega) = \frac{\rho_1(1 - z^{-1})}{1 - \rho_1^2 z^{-1}} = \frac{\rho_1(1 - e^{-j\omega T})}{1 - \rho_1^2 e^{-j\omega T}}$$

where we used $\eta = \eta_0/n = \eta_0/2$. Noting that $\omega T = \omega_0 T(1+\delta) = 2\pi(1+\delta)$, we can expand the delay exponential to first-order in δ:

$$z^{-1} = e^{-j\omega T} = e^{-2\pi j(1+\delta)} = e^{-2\pi j}e^{-2\pi j\delta} = e^{-2\pi j\delta} \simeq 1 - 2\pi j\delta$$

Thus, the reflection response becomes to first-order in δ:

$$\Gamma_1 \simeq \frac{\rho_1\big(1-(1-2\pi j\delta)\big)}{1-\rho_1^2(1-2\pi j\delta)} = \frac{\rho_1 2\pi j\delta}{1-\rho_1^2+\rho_1^2 2\pi j\delta} \simeq \frac{\rho_1 2\pi j\delta}{1-\rho_1^2}$$

where we replaced the denominator by its zeroth-order approximation because the numerator is already first-order in δ. It follows that the power reflection response will be:

$$|\Gamma_1|^2 = \frac{\rho_1^2(2\pi\delta)^2}{(1-\rho_1^2)^2}$$

Evaluating this expression for $\delta = 0.01$ and $\rho_1 = -1/3$, we find $|\Gamma|^2 = 0.00049$, or 0.049 percent of the incident power gets reflected. Next, we find the frequency about ω_0 at which the reflected power is $A = 30$ dB below the incident power. Writing again, $\omega = \omega_0 + \delta\omega = \omega_0(1+\delta)$ and assuming δ is small, we have the condition:

$$|\Gamma_1|^2 = \frac{\rho_1^2(2\pi\delta)^2}{(1-\rho_1^2)^2} = \frac{\mathcal{P}_{\text{refl}}}{\mathcal{P}_{\text{inc}}} = 10^{-A/10} \quad\Rightarrow\quad \delta = \frac{1-\rho_1^2}{2\pi|\rho_1|}\,10^{-A/20}$$

Evaluating this expression, we find $\delta = 0.0134$, or $\delta\omega = 0.0134\omega_0$. The bandwidth will be twice that, $\Delta\omega = 2\delta\omega = 0.0268\omega_0$, or in Hz, $\Delta f = 0.0268 f_0 = 268$ MHz. □

Example 4.5.5: Because of manufacturing imperfections, suppose that the actual constructed thickness of the above radome is 1% off the desired half-wavelength thickness. Determine the percentage of reflected power in this case.

Solution: This is essentially the same as the previous example. Indeed, the quantity $\theta = \omega T = 2k_1 l_1 = 2\omega l_1/c_1$ can change either because of ω or because of l_1. A simultaneous infinitesimal change (about the nominal value $\theta_0 = \omega_0 T = 2\pi$) will give:

$$\delta\theta = 2(\delta\omega)l_1/c_1 + 2\omega_0(\delta l_1)/c_1 \quad\Rightarrow\quad \delta = \frac{\delta\theta}{\theta_0} = \frac{\delta\omega}{\omega_0} + \frac{\delta l_1}{l_1}$$

In the previous example, we varied ω while keeping l_1 constant. Here, we vary l_1, while keeping ω constant, so that $\delta = \delta l_1/l_1$. Thus, we have $\delta\theta = \theta_0\delta = 2\pi\delta$. The corresponding delay factor becomes approximately $z^{-1} = e^{-j\theta} = e^{-j(2\pi+\delta\theta)} = 1 - j\delta\theta = 1 - 2\pi j\delta$. The resulting expression for the power reflection response is identical to the above and its numerical value is the same if $\delta = 0.01$. □

Example 4.5.6: Because of weather conditions, suppose that the characteristic impedance of the medium outside the above radome is 1% off the impedance inside. Calculate the percentage of reflected power in this case.

Solution: Suppose that the outside impedance changes to $\eta_b = \eta_0 + \delta\eta$. The wave impedance at the outer interface will be $Z_2 = \eta_b = \eta_0 + \delta\eta$. Because the slab length is still a half-wavelength, the wave impedance at the inner interface will be $Z_1 = Z_2 = \eta_0 + \delta\eta$. It follows that the reflection response will be:

$$\Gamma_1 = \frac{Z_1 - \eta_0}{Z_1 + \eta_0} = \frac{\eta_0 + \delta\eta - \eta_0}{\eta_0 + \delta\eta + \eta_0} = \frac{\delta\eta}{2\eta_0 + \delta\eta} \simeq \frac{\delta\eta}{2\eta_0}$$

where we replaced the denominator by its zeroth-order approximation in $\delta\eta$. Evaluating at $\delta\eta/\eta_0 = 1\% = 0.01$, we find $\Gamma_1 = 0.005$, which leads to a reflected power of $|\Gamma_1|^2 = 2.5\times10^{-5}$, or, 0.0025 percent. □

4.6 Time-Domain Reflection Response

We conclude our discussion of the single slab by trying to understand its behavior in the time domain. The z-domain reflection transfer function of Eq. (4.4.5) incorporates the effect of all multiple reflections that are set up within the slab as the wave bounces back and forth at the left and right interfaces. Expanding Eq. (4.4.5) in a partial fraction expansion and then in power series in z^{-1} gives:

$$\Gamma_1(z) = \frac{\rho_1 + \rho_2 z^{-1}}{1+\rho_1\rho_2 z^{-1}} = \frac{1}{\rho_1} - \frac{1}{\rho_1}\frac{(1-\rho_1^2)}{1+\rho_1\rho_2 z^{-1}} = \rho_1 + \sum_{n=1}^{\infty}(1-\rho_1^2)(-\rho_1)^{n-1}\rho_2^n z^{-n}$$

Using the reflection coefficient from the right of the first interface, $\rho_1' = -\rho_1$, and the transmission coefficients $\tau_1 = 1+\rho_1$ and $\tau_1' = 1+\rho_1' = 1-\rho_1$, we have $\tau_1\tau_1' = 1-\rho_1^2$. Then, the above power series can be written as a function of frequency in the form:

$$\Gamma_1(\omega) = \rho_1 + \sum_{n=1}^{\infty}\tau_1\tau_1'(\rho_1')^{n-1}\rho_2^n z^{-n} = \rho_1 + \sum_{n=1}^{\infty}\tau_1\tau_1'(\rho_1')^{n-1}\rho_2^n e^{-j\omega nT}$$

where we set $z^{-1} = e^{-j\omega T}$. It follows that the time-domain *reflection impulse response*, that is, the inverse Fourier transform of $\Gamma_1(\omega)$, will be the sum of discrete impulses:

$$\Gamma_1(t) = \rho_1\delta(t) + \sum_{n=1}^{\infty}\tau_1\tau_1'(\rho_1')^{n-1}\rho_2^n\,\delta(t-nT) \tag{4.6.1}$$

This is the response of the slab to a forward-moving impulse striking the left interface at $t = 0$, that is, the response to the input $E_{1+}(t) = \delta(t)$. The first term $\rho_1\delta(t)$ is the impulse immediately reflected at $t = 0$ with the reflection coefficient ρ_1. The remaining terms represent the multiple reflections within the slab. Fig. 4.6.1 is a *timing diagram* that traces the reflected and transmitted impulses at the first and second interfaces.

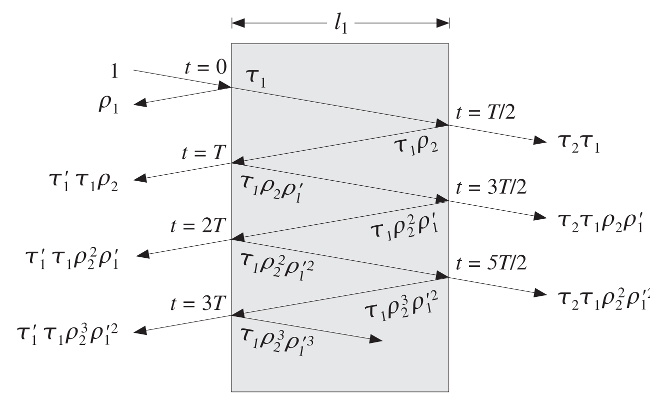


Fig. 4.6.1 Multiple reflections building up the reflection and transmission responses.

The input pulse $\delta(t)$ gets transmitted to the inside of the left interface and picks up a transmission coefficient factor τ_1. In $T/2$ seconds this pulse strikes the right interface and causes a reflected wave whose amplitude is changed by the reflection coefficient ρ_2 into $\tau_1\rho_2$.

Thus, the pulse $\tau_1\rho_2\delta(t-T/2)$ gets reflected backwards and will arrive at the left interface $T/2$ seconds later, that is, at time $t=T$. A proportion τ_1' of it will be transmitted through to the left, and a proportion ρ_1' will be re-reflected towards the right. Thus, at time $t=T$, the transmitted pulse into the left medium will be $\tau_1\tau_1'\rho_2\delta(t-T)$, and the re- reflected pulse $\tau_1\rho_1'\rho_2\delta(t-T)$.

The re-reflected pulse will travel forward to the right interface, arriving there at time $t=3T/2$ getting reflected backwards picking up a factor ρ_2. This will arrive at the left at time $t=2T$. The part transmitted to the left will be now $\tau_1\tau_1'\rho_1'\rho_2^2\delta(t-2T)$, and the part re-reflected to the right $\tau_1\rho_1'^2\rho_2^2\delta(t-2T)$. And so on, after the nth round trip, the pulse transmitted to the left will be $\tau_1\tau_1'(\rho_1')^{n-1}\rho_2^n\delta(t-nT)$. The sum of all the reflected pulses will be $\Gamma_1(t)$ of Eq. (4.6.1).

In a similar way, we can derive the overall *transmission response* to the right. It is seen in the figure that the transmitted pulse at time $t=nT+(T/2)$ will be $\tau_1\tau_2(\rho_1')^n\rho_2^n$. Thus, the overall transmission impulse response will be:

$$\mathcal{T}(t)=\sum_{n=0}^{\infty}\tau_1\tau_2(\rho_1')^n\rho_2^n\,\delta(t-nT-T/2)$$

It follows that its Fourier transform will be:

$$\mathcal{T}(\omega)=\sum_{n=0}^{\infty}\tau_1\tau_2(\rho_1')^n\rho_2^n e^{-jn\omega T}e^{-j\omega T/2}$$

which sums up to Eq. (4.4.6):

$$\mathcal{T}(\omega)=\frac{\tau_1\tau_2 e^{-j\omega T/2}}{1-\rho_1'\rho_2 e^{-j\omega T}}=\frac{\tau_1\tau_2 e^{-j\omega T/2}}{1+\rho_1\rho_2 e^{-j\omega T}} \tag{4.6.2}$$

For an incident field $E_{1+}(t)$ with arbitrary time dependence, the overall reflection response of the slab is obtained by convolving the impulse response $\Gamma_1(t)$ with $E_{1+}(t)$. This follows from the linear superposition of the reflection responses of all the frequency components of $E_{1+}(t)$, that is,

$$E_{1-}(t)=\int_{-\infty}^{\infty}\Gamma_1(\omega)E_{1+}(\omega)e^{j\omega t}\frac{d\omega}{2\pi}\,,\quad\text{where}\quad E_{1+}(t)=\int_{-\infty}^{\infty}E_{1+}(\omega)e^{j\omega t}\frac{d\omega}{2\pi}$$

Then, the convolution theorem of Fourier transforms implies that:

$$E_{1-}(t)=\int_{-\infty}^{\infty}\Gamma_1(\omega)E_{1+}(\omega)e^{j\omega t}\frac{d\omega}{2\pi}=\int_{-\infty}^{-\infty}\Gamma_1(t')E_{1+}(t-t')\,dt' \tag{4.6.3}$$

Inserting (4.6.1), we find that the reflected wave arises from the multiple reflections of $E_{1+}(t)$ as it travels and bounces back and forth between the two interfaces:

$$E_{1-}(t)=\rho_1E_{1+}(t)+\sum_{n=1}^{\infty}\tau_1\tau_1'(\rho_1')^{n-1}\rho_2^n\,E_{1+}(t-nT) \tag{4.6.4}$$

For a causal waveform $E_{1+}(t)$, the summation over n will be finite, such that at each time $t\geq 0$ only the terms that have $t-nT\geq 0$ will be present. In a similar fashion, we find for the overall transmitted response into medium η_b:

$$E_{2+}'(t)=\int_{-\infty}^{-\infty}\mathcal{T}(t')E_{1+}(t-t')\,dt'=\sum_{n=0}^{\infty}\tau_1\tau_2(\rho_1')^n\rho_2^n\,E_{1+}(t-nT-T/2) \tag{4.6.5}$$

We will use similar techniques later on to determine the transient responses of transmission lines.

4.7 Two Dielectric Slabs

Next, we consider more than two interfaces. As we mentioned in the previous section, Eqs. (4.4.7)–(4.4.9) are general and can be applied to all successive interfaces. Fig. 4.7.1 shows three interfaces separating four media. The overall reflection response can be calculated by successive application of Eq. (4.4.8):

$$\Gamma_1=\frac{\rho_1+\Gamma_2e^{-2jk_1l_1}}{1+\rho_1\Gamma_2e^{-2jk_1l_1}}\,,\quad \Gamma_2=\frac{\rho_2+\Gamma_3e^{-2jk_2l_2}}{1+\rho_2\Gamma_3e^{-2jk_2l_2}}$$

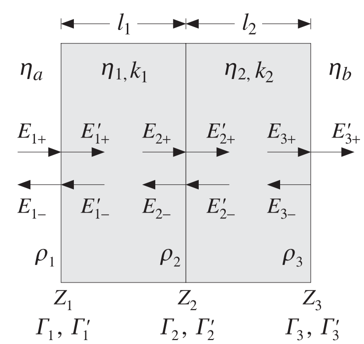


Fig. 4.7.1 Two dielectric slabs.

If there is no backward-moving wave in the right-most medium, then $\Gamma_3'=0$, which implies $\Gamma_3=\rho_3$. Substituting Γ_2 into Γ_1 and denoting $z_1=e^{2jk_1l_1}$, $z_2=e^{2jk_2l_2}$, we eventually find:

$$\Gamma_1=\frac{\rho_1+\rho_2z_1^{-1}+\rho_1\rho_2\rho_3z_2^{-1}+\rho_3z_1^{-1}z_2^{-1}}{1+\rho_1\rho_2z_1^{-1}+\rho_2\rho_3z_2^{-1}+\rho_1\rho_3z_1^{-1}z_2^{-1}} \tag{4.7.1}$$

The reflection response Γ_1 can alternatively be determined from the knowledge of the wave impedance $Z_1=E_1/H_1$ at interface-1:

$$\Gamma_1=\frac{Z_1-\eta_a}{Z_1+\eta_a}$$

The fields E_1, H_1 are obtained by successively applying Eq. (4.4.9):

$$\begin{bmatrix} E_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} \cos k_1 l_1 & j\eta_1 \sin k_1 l_1 \\ j\eta_1^{-1} \sin k_1 l_1 & \cos k_1 l_1 \end{bmatrix} \begin{bmatrix} E_2 \\ H_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos k_1 l_1 & j\eta_1 \sin k_1 l_1 \\ j\eta_1^{-1} \sin k_1 l_1 & \cos k_1 l_1 \end{bmatrix} \begin{bmatrix} \cos k_2 l_2 & j\eta_2 \sin k_2 l_2 \\ j\eta_2^{-1} \sin k_2 l_2 & \cos k_2 l_2 \end{bmatrix} \begin{bmatrix} E_3 \\ H_3 \end{bmatrix}$$

But at interface-3, $E_3 = E'_3 = E'_{3+}$ and $H_3 = Z_3^{-1} E_3 = \eta_b^{-1} E'_{3+}$, because $Z_3 = \eta_b$. Therefore, we can obtain the fields E_1, H_1 by the matrix multiplication:

$$\begin{bmatrix} E_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} \cos k_1 l_1 & j\eta_1 \sin k_1 l_1 \\ j\eta_1^{-1} \sin k_1 l_1 & \cos k_1 l_1 \end{bmatrix} \begin{bmatrix} \cos k_2 l_2 & j\eta_2 \sin k_2 l_2 \\ j\eta_2^{-1} \sin k_2 l_2 & \cos k_2 l_2 \end{bmatrix} \begin{bmatrix} 1 \\ \eta_b^{-1} \end{bmatrix} E'_{3+}$$

Because Z_1 is the ratio of E_1 and H_1, the factor E'_{3+} cancels out and can be set equal to unity.

Example 4.7.1: Determine Γ_1 if both slabs are quarter-wavelength slabs. Repeat if both slabs are half-wavelength and when one is half- and the other quarter-wavelength.

Solution: Because $l_1 = \lambda_1/4$ and $l_2 = \lambda_2/4$, we have $2k_1 l_1 = 2k_2 l_2 = \pi$, and it follows that $z_1 = z_2 = -1$. Then, Eq. (4.7.1) becomes:

$$\Gamma_1 = \frac{\rho_1 - \rho_2 - \rho_1\rho_2\rho_3 + \rho_3}{1 - \rho_1\rho_2 - \rho_2\rho_3 + \rho_1\rho_3}$$

A simpler approach is to work with wave impedances. Using $Z_3 = \eta_b$, we have:

$$Z_1 = \frac{\eta_1^2}{Z_2} = \frac{\eta_1^2}{\eta_2^2/Z_3} = \frac{\eta_1^2}{\eta_2^2} Z_3 = \frac{\eta_1^2}{\eta_2^2}\eta_b$$

Inserting this into $\Gamma_1 = (Z_1 - \eta_a)/(Z_1 + \eta_a)$, we obtain:

$$\Gamma_1 = \frac{\eta_1^2\eta_b - \eta_2^2\eta_a}{\eta_1^2\eta_b + \eta_2^2\eta_a}$$

The two expressions for Γ_1 are equivalent. The input impedance Z_1 can also be obtained by matrix multiplication. Because $k_1 l_1 = k_2 l_2 = \pi/2$, we have $\cos k_1 l_1 = 0$ and $\sin k_1 l_1 = 1$ and the propagation matrices for E_1, H_1 take the simplified form:

$$\begin{bmatrix} E_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} 0 & j\eta_1 \\ j\eta_1^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & j\eta_2 \\ j\eta_2^{-1} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \eta_b^{-1} \end{bmatrix} E'_{3+} = \begin{bmatrix} -\eta_1\eta_2^{-1} \\ -\eta_2\eta_1^{-1}\eta_b^{-1} \end{bmatrix} E'_{3+}$$

The ratio E_1/H_1 gives the same answer for Z_1 as above. When both slabs are half-wavelength, the impedances propagate unchanged: $Z_1 = Z_2 = Z_3$, but $Z_3 = \eta_b$.

If η_1 is half- and η_2 quarter-wavelength, then, $Z_1 = Z_2 = \eta_2^2/Z_3 = \eta_2^2/\eta_b$. And, if the quarter-wavelength is first and the half-wavelength second, $Z_1 = \eta_1^2/Z_2 = \eta_1^2/Z_3 = \eta_1^2/\eta_b$. The corresponding reflection coefficient Γ_1 is in the three cases:

$$\Gamma_1 = \frac{\eta_b - \eta_a}{\eta_b + \eta_a}, \quad \Gamma_1 = \frac{\eta_2^2 - \eta_a\eta_b}{\eta_2^2 + \eta_a\eta_b}, \quad \Gamma_1 = \frac{\eta_1^2 - \eta_a\eta_b}{\eta_1^2 + \eta_a\eta_b}$$

These expressions can also be derived by Eq. (4.7.1), or by the matrix method. □

The frequency dependence of Eq. (4.7.1) arises through the factors z_1, z_2, which can be written in the forms: $z_1 = e^{j\omega T_1}$ and $z_2 = e^{j\omega T_2}$, where $T_1 = 2l_1/c_1$ and $T_2 = 2l_2/c_2$ are the two-way travel time delays through the two slabs.

A case of particular interest arises when the slabs are designed to have the *equal* travel-time delays so that $T_1 = T_2 \equiv T$. Then, defining a common variable $z = z_1 = z_2 = e^{j\omega T}$, we can write the reflection response as a second-order digital filter transfer function:

$$\Gamma_1(z) = \frac{\rho_1 + \rho_2(1 + \rho_1\rho_3)z^{-1} + \rho_3 z^{-2}}{1 + \rho_2(\rho_1 + \rho_3)z^{-1} + \rho_1\rho_3 z^{-2}} \tag{4.7.2}$$

In the next chapter, we discuss further the properties of such higher-order reflection transfer functions arising from multilayer dielectric slabs.

4.8 Reflection by a Moving Boundary

Reflection and transmission by moving boundaries, such as reflection from a moving mirror, introduce Doppler shifts in the frequencies of the reflected and transmitted waves. Here, we look at the problem of normal incidence on a dielectric interface that is moving with constant velocity v perpendicularly to the interface, that is, along the z-direction as shown in Fig. 4.8.1. Additional examples may be found in [123–140]. The case of oblique incidence is discussed in Sec. 6.8.

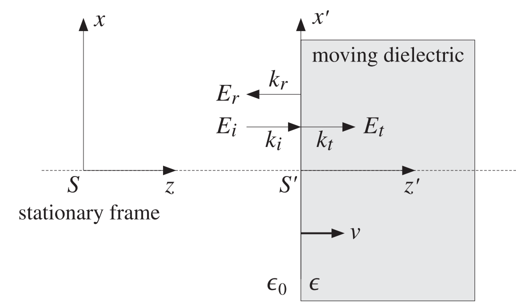


Fig. 4.8.1 Reflection and transmission at a moving boundary.

The dielectric is assumed to be non-magnetic and lossless with permittivity ϵ. The left medium is free space ϵ_0. The electric field is assumed to be in the x-direction and thus, the magnetic field will be in the y-direction. We consider two coordinate frames, the fixed frame S with coordinates $\{t, x, y, z\}$, and the moving frame S' with $\{t', x', y', z'\}$. The two sets of coordinates are related by the Lorentz transformation equations (G.1) of Appendix G.

We are interested in determining the Doppler-shifted frequencies of the reflected and transmitted waves, as well as the reflection and transmission coefficients as measured in the fixed frame S.

The procedure for solving this type of problem—originally suggested by Einstein in his 1905 special relativity paper [123]—is to solve the reflection and transmission problem in the moving frame S' with respect to which the boundary is at rest, and then transform the results back to the fixed frame S using the Lorentz transformation properties of the fields. In the fixed frame S, the fields to the left and right of the interface will have the forms:

$$\text{left}\begin{cases} E_x = E_i e^{j(\omega t-k_i z)} + E_r e^{j(\omega_r t+k_r z)} \\ H_y = H_i e^{j(\omega t-k_i z)} - H_r e^{j(\omega_r t+k_r z)}\end{cases} \qquad \text{right}\begin{cases} E_x = E_t e^{j(\omega_t t-k_t z)} \\ H_y = H_t e^{j(\omega_t t-k_t z)}\end{cases} \tag{4.8.1}$$

where $\omega, \omega_r, \omega_t$ and k_i, k_r, k_t are the frequencies and wavenumbers of the incident, reflected, and transmitted waves measured in S. Because of Lorentz invariance, the propagation phases remain unchanged in the frames S and S', that is,

$$\begin{aligned} \phi_i &= \omega t - k_i z = \omega' t' - k_i' z' = \phi_i' \\ \phi_r &= \omega_r t + k_r z = \omega' t' + k_r' z' = \phi_r' \\ \phi_t &= \omega_t t - k_t z = \omega' t' - k_t' z' = \phi_t' \end{aligned} \tag{4.8.2}$$

In the frame S' where the dielectric is at rest, all three frequencies are the same and set equal to ω'. This is a consequence of the usual tangential boundary conditions applied to the interface at rest. Note that ϕ_r can be written as $\phi_r = \omega_r t - (-k_r)z$ implying that the reflected wave is propagating in the negative z-direction. In the rest frame S' of the boundary, the wavenumbers are:

$$k_i' = \frac{\omega'}{c}, \quad k_r' = \frac{\omega'}{c}, \quad k_t' = \omega'\sqrt{\epsilon\mu_0} = n\frac{\omega'}{c} \tag{4.8.3}$$

where c is the speed of light in vacuum and $n = \sqrt{\epsilon/\epsilon_0}$ is the refractive index of the dielectric at rest. The frequencies and wavenumbers in the fixed frame S are related to those in S' by applying the Lorentz transformation of Eq. (G.14) to the frequency-wavenumber four-vectors $(\omega/c, 0, 0, k_i)$, $(\omega_r/c, 0, 0, -k_r)$, and $(\omega_t/c, 0, 0, k_t)$:

$$\begin{aligned} \omega &= \gamma(\omega' + \beta c k_i') = \omega'\gamma(1+\beta) \\ k_i &= \gamma(k_i' + \frac{\beta}{c}\omega') = \frac{\omega'}{c}\gamma(1+\beta) \\ \omega_r &= \gamma\big(\omega' + \beta c(-k_r')\big) = \omega'\gamma(1-\beta) \\ -k_r &= \gamma(-k_r' + \frac{\beta}{c}\omega') = -\frac{\omega'}{c}\gamma(1-\beta) \\ \omega_t &= \gamma(\omega' + \beta c k_t') = \omega'\gamma(1+\beta n) \\ k_t &= \gamma(k_t' + \frac{\beta}{c}\omega') = \frac{\omega'}{c}\gamma(n+\beta) \end{aligned} \tag{4.8.4}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. Eliminating the primed quantities, we obtain the Doppler-shifted frequencies of the reflected and transmitted waves:

$$\boxed{\omega_r = \omega\frac{1-\beta}{1+\beta}}, \qquad \boxed{\omega_t = \omega\frac{1+\beta n}{1+\beta}} \tag{4.8.5}$$

The phase velocities of the incident, reflected, and transmitted waves are:

$$v_i = \frac{\omega}{k_i} = c, \quad v_r = \frac{\omega_r}{k_r} = c, \quad v_t = \frac{\omega_t}{k_t} = c\frac{1+\beta n}{n+\beta} \tag{4.8.6}$$

These can also be derived by applying Einstein's velocity addition theorem of Eq. (G.8). For example, we have for the transmitted wave:

$$v_t = \frac{v_d + v}{1 + v_d v/c^2} = \frac{c/n + v}{1 + (c/n)v/c^2} = c\frac{1+\beta n}{n+\beta}$$

where $v_d = c/n$ is the phase velocity within the dielectric at rest. To first-order in $\beta = v/c$, the phase velocity within the moving dielectric becomes:

$$v_t = c\frac{1+\beta n}{n+\beta} \simeq \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right)$$

The second term is known as the "Fresnel drag." The quantity $n_t = (n+\beta)/(1+\beta n)$ may be thought of as the "effective" refractive index of the moving dielectric as measured in the fixed system S.

Next, we derive the reflection and transmission coefficients. In the rest-frame S' of the dielectric, the fields have the usual forms derived earlier in Sections 4.1 and 4.2:

$$\text{left}\begin{cases} E_x' = E_i'(e^{j\phi_i'} + \rho e^{j\phi_r'}) \\ H_y' = \dfrac{1}{\eta_0}E_i'(e^{j\phi_i'} - \rho e^{j\phi_r'})\end{cases} \qquad \text{right}\begin{cases} E_x' = \tau E_i' e^{j\phi_t'} \\ H_y' = \dfrac{1}{\eta}\tau E_i' e^{j\phi_t'}\end{cases} \tag{4.8.7}$$

where

$$\eta = \frac{\eta_0}{n}, \quad \rho = \frac{\eta - \eta_0}{\eta + \eta_0} = \frac{1-n}{1+n}, \quad \tau = 1 + \rho = \frac{2}{1+n}$$

The primed fields can be transformed to the fixed frame S using the inverse of the Lorentz transformation equations (G.31), that is,

$$\begin{aligned} E_x &= \gamma(E_x' + \beta c B_y') = \gamma(E_x' + \beta\eta_0 H_y') \\ H_y &= \gamma(H_y' + c\beta D_x') = \gamma(H_y' + c\beta\epsilon E_x') \end{aligned} \tag{4.8.8}$$

where we replaced $B_y' = \mu_0 H_y'$, $c\mu_0 = \eta_0$, and $D_x' = \epsilon E_x'$ (of course, $\epsilon = \epsilon_0$ in the left medium). Using the invariance of the propagation phases, we find for the fields at the left side of the interface:

$$E_x = \gamma\big[E_i'(e^{j\phi_i} + \rho e^{j\phi_r}) + \beta E_i'(e^{j\phi_i} - \rho e^{j\phi_r})\big] = E_i'\gamma\big[(1+\beta)e^{j\phi_i} + \rho(1-\beta)e^{j\phi_r}\big] \tag{4.8.9}$$

Similarly, for the right side of the interface we use the property $\eta_0/\eta = n$ to get:

$$E_x = \gamma\big[\tau E_i' e^{j\phi_t} + \beta n\tau E_i' e^{j\phi_t}\big] = \gamma\tau E_i'(1+\beta n)e^{j\phi_t} \tag{4.8.10}$$

Comparing these with Eq. (4.8.1), we find the incident, reflected, and transmitted electric field amplitudes:

$$E_i = \gamma E_i'(1+\beta), \quad E_r = \rho\gamma E_i'(1-\beta), \quad E_t = \tau\gamma E_i'(1+\beta n) \tag{4.8.11}$$

from which we obtain the reflection and transmission coefficients in the fixed frame S:

$$\boxed{\frac{E_r}{E_i} = \rho\frac{1-\beta}{1+\beta}}\,, \qquad \boxed{\frac{E_t}{E_i} = \tau\frac{1+\beta n}{1+\beta}} \tag{4.8.12}$$

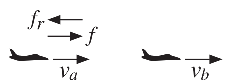
The case of a perfect mirror is also covered by these expressions by setting $\rho = -1$ and $\tau = 0$. Eq. (4.8.5) is widely used in Doppler radar applications. Typically, the boundary (the target) is moving at non-relativistic speeds so that $\beta = v/c \ll 1$. In such case, the first-order approximation of (4.8.5) is adequate:

$$f_r \simeq f(1-2\beta) = f(1-2\frac{v}{c}) \quad \Rightarrow \quad \frac{\Delta f}{f} = -2\frac{v}{c} \tag{4.8.13}$$

where $\Delta f = f_r - f$ is the Doppler shift. The negative sign means that $f_r < f$ if the target is receding away from the source of the wave, and $f_r > f$ if it is approaching the source.

As we mentioned in Sec. 2.11, if the source of the wave is moving with velocity v_a and the target with velocity v_b (with respect to a common fixed frame, such as the ground), then one must use the relative velocity $v = v_b - v_a$ in the above expression:

$$\boxed{\frac{\Delta f}{f} = \frac{f_r - f}{f} = 2\frac{v_a - v_b}{c}} \tag{4.8.14}$$



4.9 Problems

4.1 Fill in the details of the equivalence between Eq. (4.2.2) and (4.2.3), that is,

$$\begin{aligned} E_+ + E_- &= E'_+ + E'_- \\ \frac{1}{\eta}(E_+ - E_-) &= \frac{1}{\eta'}(E'_+ - E'_-) \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \frac{1}{\tau}\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\begin{bmatrix} E'_+ \\ E'_- \end{bmatrix}$$

4.2 Fill in the details of the equivalences stated in Eq. (4.2.9), that is,

$$Z = Z' \quad \Leftrightarrow \quad \Gamma = \frac{\rho + \Gamma'}{1 + \rho\Gamma'} \quad \Leftrightarrow \quad \Gamma' = \frac{\rho' + \Gamma}{1 + \rho'\Gamma}$$

Show that if there is no left-incident field from the right, then $\Gamma = \rho$, and if there is no right-incident field from the left, then, $\Gamma' = 1/\rho'$. Explain the asymmetry of the two cases.

4.3 Let ρ, τ be the reflection and transmission coefficients from the left side of an interface and let ρ', τ' be those from the right, as defined in Eq. (4.2.5). One of the two media may be lossy, and therefore, its characteristic impedance and hence ρ, τ may be complex-valued. Show and interpret the relationships:

$$1 - |\rho|^2 = \text{Re}(\frac{\eta}{\eta'})|\tau|^2 = \text{Re}(\tau^*\tau')$$

4.4 Show that the reflection and transmission responses of the single dielectric slab of Fig. 4.4.1 are given by Eq. (4.4.6), that is,

$$\Gamma = \frac{\rho_1 + \rho_2 e^{-2jk_1l_1}}{1 + \rho_1\rho_2 e^{-2jk_1l_1}}\,, \qquad \mathcal{T} = \frac{E'_{2+}}{E_{1+}} = \frac{\tau_1\tau_2 e^{-jk_1l_1}}{1 + \rho_1\rho_2 e^{-2jk_1l_1}}$$

Moreover, using these expressions show and interpret the relationship:

$$\frac{1}{\eta_a}(1 - |\Gamma|^2) = \frac{1}{\eta_b}|\mathcal{T}|^2$$

4.5 A 1-GHz plane wave is incident normally onto a thick copper plate ($\sigma = 5.8\times10^7$ S/m.) Can the plate be considered to be a good conductor at this frequency? Calculate the percentage of the incident power that enters the plate. Calculate the attenuation coefficient within the conductor and express it in units of dB/m. What is the penetration depth in mm?

4.6 With the help of Fig. 4.5.1, argue that the 3-dB width $\Delta\omega$ is related to the 3-dB frequency ω_3 by $\Delta\omega = 2\omega_3$ and $\Delta\omega = \omega_0 - 2\omega_3$, in the cases of half- and quarter-wavelength slabs. Then, show that ω_3 and $\Delta\omega$ are given by:

$$\cos\omega_3 T = \pm\frac{2\rho_1^2}{1+\rho_1^4}\,, \quad \tan\left(\frac{\Delta\omega T}{4}\right) = \frac{1-\rho_1^2}{1+\rho_1^2}$$

4.7 A fiberglass ($\epsilon = 4\epsilon_0$) radome protecting a microwave antenna is designed as a half-wavelength reflectionless slab at the operating frequency of 12 GHz.

a. Determine three possible thicknesses (in cm) for this radome.

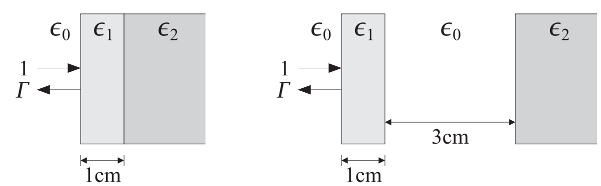
b. Determine the 15-dB and 30-dB bandwidths in GHz about the 12 GHz operating frequency , defined as the widths over which the reflected power is 15 or 30 dB below the incident power.

4.8 A 5 GHz wave is normally incident from air onto a dielectric slab of thickness of 1 cm and refractive index of 1.5, as shown below. The medium to the right of the slab has an index of 2.25.

a. Write an analytical expression of the reflectance $|\Gamma(f)|^2$ as a function of frequency and sketch it versus f over the interval $0 \le f \le 15$ GHz. What is the value of the reflectance at 5 GHz?

b. Next, the 1-cm slab is moved to the left by a distance of 3 cm, creating an air-gap between it and the rightmost dielectric. Repeat all the questions of part (a).

c. Repeat part (a), if the slab thickness is 2 cm.



4.9 Consider a two-layer dielectric structure as shown in Fig. 4.7.1, and let n_a, n_1, n_2, n_b be the refractive indices of the four media. Consider the four cases: (a) both layers are quarter-wave, (b) both layers are half-wave, (c) layer-1 is quarter- and layer-2 half-wave, and (d) layer-1 is half- and layer-2 quarter-wave. Show that the reflection coefficient at interface-1 is given by the following expressions in the four cases:

$$\Gamma_1 = \frac{n_a n_2^2 - n_b n_1^2}{n_a n_2^2 + n_b n_1^2}\,, \quad \Gamma_1 = \frac{n_a - n_b}{n_a + n_b}\,, \quad \Gamma_1 = \frac{n_a n_b - n_1^2}{n_a n_b + n_1^2}\,, \quad \Gamma_1 = \frac{n_a n_b - n_2^2}{n_a n_b + n_2^2}$$

4.10 Consider the lossless two-slab structure of Fig. 4.7.1. Write down all the transfer matrices relating the fields $E_{i\pm}$, $i = 1, 2, 3$ at the left sides of the three interfaces. Then, show the energy conservation equations:

$$\frac{1}{\eta_a}(|E_{1+}|^2 - |E_{1-}|^2) = \frac{1}{\eta_1}(|E_{2+}|^2 - |E_{2-}|^2) = \frac{1}{\eta_2}(|E_{3+}|^2 - |E_{3-}|^2) = \frac{1}{\eta_b}|E'_{1+}|^2$$

4.11 An alternative way of representing the propagation relationship Eq. (4.1.12) is in terms of the hyperbolic w-plane variable defined in terms of the reflection coefficient Γ, or equivalently, the wave impedance Z as follows:

$$\Gamma = e^{-2w} \quad \Leftrightarrow \quad Z = \eta \coth(w) \tag{4.9.1}$$

Show the equivalence of these expressions. Writing $\Gamma_1 = e^{-2w_1}$ and $\Gamma_2 = e^{-2w_2}$, show that Eq. (4.1.12) becomes equivalent to:

$$w_1 = w_2 + jkl \qquad \text{(propagation in } w\text{-domain)} \tag{4.9.2}$$

This form is essentially the mathematical (as opposed to graphical) version of the *Smith chart* and is particularly useful for numerical computations using MATLAB.

4.12 A fighter plane flying at a speed of 900 km/hr with respect to the ground is closing in on a target aircraft. The fighter's Doppler radar, operating at the X-band frequency of 10 GHz, detects a positive Doppler shift of 2 kHz in the return frequency. Determine the speed of the target with respect to the ground. [*Ans.* 792 km/hr.]

4.13 The complete set of Lorentz transformations of the fields in Eq. (4.8.8) is as follows (see also Eq. (G.31) of Appendix G):

$$E_x = \gamma(E'_x + \beta c B'_y), \quad H_y = \gamma(H'_y + c\beta D'_x), \quad D_x = \gamma(D'_x + \frac{1}{c}\beta H'_y), \quad B_y = \gamma(B'_y + \frac{1}{c}\beta E'_x)$$

The constitutive relations in the rest frame S' of the moving dielectric are the usual ones, that is, $B'_y = \mu H'_y$ and $D'_x = \epsilon E'_x$. By eliminating the primed quantities in terms of the unprimed ones, show that the constitutive relations have the following form in the fixed system S:

$$D_x = \frac{(1-\beta^2)\epsilon E_x - (n^2-1)H_y/c}{1-\beta^2 n^2}, \quad B_y = \frac{(1-\beta^2)\mu H_y - (n^2-1)E_x/c}{1-\beta^2 n^2}$$

where n is the refractive index of the moving medium, $n = \sqrt{\epsilon\mu/\epsilon_o\mu_0}$. Show that for free space, the constitutuve relations remain the same as in the frame S'.