

## 9

## Transmission Lines

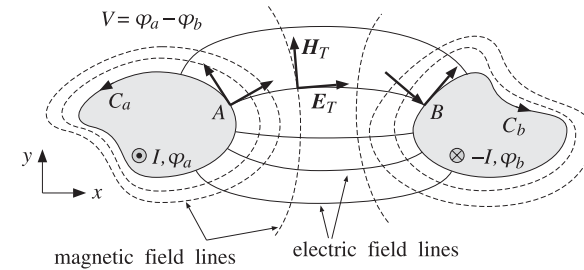


Fig. 9.1.1 Two-conductor transmission line.

## 9.1 General Properties of TEM Transmission Lines

We saw in Sec. 8.3 that TEM modes are described by Eqs. (8.3.3) and (8.3.4), the latter being equivalent to a two-dimensional electrostatic problem:

$$\begin{cases} \mathbf{H}_T = \frac{1}{\eta} \hat{\mathbf{z}} \times \mathbf{E}_T \\ \nabla_T \times \mathbf{E}_T = 0 \\ \nabla_T \cdot \mathbf{E}_T = 0 \end{cases} \quad (\text{TEM modes}) \quad (9.1.1)$$

The second of (9.1.1) implies that  $\mathbf{E}_T$  can be expressed as the (two-dimensional) gradient of a scalar electrostatic potential. Then, the third equation becomes Laplace's equation for the potential. Thus, the electric field can be obtained from:

$$\begin{cases} \nabla_T^2 \varphi = 0 \\ \mathbf{E}_T = -\nabla_T \varphi \end{cases} \quad (\text{equivalent electrostatic problem}) \quad (9.1.2)$$

Because in electrostatic problems the electric field lines must start at positively charged conductors and end at negatively charged ones, a TEM mode can be supported only in multi-conductor guides, such as the coaxial cable or the two-wire line. Hollow conducting waveguides cannot support TEM modes.

Fig. 9.1.1 depicts the transverse cross-sectional area of a two-conductor transmission line. The cross-section shapes are arbitrary.

The conductors are *equipotentials* of the electrostatic solution. Let  $\varphi_a, \varphi_b$  be the constant potentials on the two conductors. The voltage difference between the conductors will be  $V = \varphi_a - \varphi_b$ . The electric field lines start perpendicularly on conductor (a) and end perpendicularly on conductor (b).

The magnetic field lines, being perpendicular to the electric lines according to Eq. (9.1.1), are recognized to be the *equipotential lines*. As such, they close upon themselves surrounding the two conductors.

In particular, on the conductor surfaces the magnetic field is tangential. According to Ampère's law, the line integrals of the magnetic field around each conductor will result into total currents  $I$  and  $-I$  flowing on the conductors in the  $z$ -direction. These currents are equal and opposite.

## Impedance, Inductance, and Capacitance

Because the fields are propagating along the  $z$ -direction with frequency  $\omega$  and wavenumber  $\beta = \omega/c$ , the  $z, t$  dependence of the voltage  $V$  and current  $I$  will be:

$$\begin{cases} V(z, t) = V e^{j\omega t - j\beta z} \\ I(z, t) = I e^{j\omega t - j\beta z} \end{cases} \quad (9.1.3)$$

For backward-moving voltage and current waves, we must replace  $\beta$  by  $-\beta$ . The ratio  $V(z, t)/I(z, t) = V/I$  remains constant and independent of  $z$ . It is called the *characteristic impedance* of the line:

$$Z = \frac{V}{I} \quad (\text{line impedance}) \quad (9.1.4)$$

In addition to the impedance  $Z$ , a TEM line is characterized by its inductance per unit length  $L'$  and its capacitance per unit length  $C'$ . For lossless lines, the three quantities  $Z, L', C'$  are related as follows:

$$L' = \mu \frac{Z}{\eta}, \quad C' = \epsilon \frac{\eta}{Z} \quad (\text{inductance and capacitance per unit length}) \quad (9.1.5)$$

where  $\eta = \sqrt{\mu/\epsilon}$  is the characteristic impedance of the dielectric medium between the conductors.<sup>†</sup> By multiplying and dividing  $L'$  and  $C'$ , we also obtain:

$$Z = \sqrt{\frac{L'}{C'}}, \quad c = \frac{1}{\sqrt{\epsilon\mu}} = \frac{1}{\sqrt{L'C'}} \quad (9.1.6)$$

<sup>†</sup>These expressions explain why  $\mu$  and  $\epsilon$  are sometimes given in units of henry/m and farad/m.

The *velocity factor* of the line is the ratio  $c/c_0 = 1/n$ , where  $n = \sqrt{\epsilon/\epsilon_0} = \sqrt{\epsilon_r}$  is the refractive index of the dielectric, which is assumed to be non-magnetic.

Because  $\omega = \beta c$ , the guide wavelength will be  $\lambda = 2\pi/\beta = c/f = c_0/fn = \lambda_0/n$ , where  $\lambda_0$  is the free-space wavelength. For a finite length  $l$  of the transmission line, the quantity  $l/\lambda = nl/\lambda_0$  is referred to as the *electrical length* of the line and plays the same role as the optical length in thin-film layers.

Eqs. (9.1.5) and (9.1.6) are general results that are valid for any TEM line. They can be derived with the help of Fig. 9.1.2.

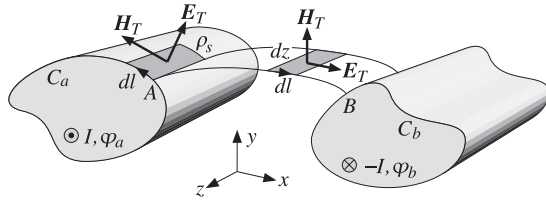


Fig. 9.1.2 Surface charge and magnetic flux linkage.

The voltage  $V$  is obtained by integrating  $E_T \cdot dl$  along any path from (a) to (b). However, if that path is chosen to be an  $E$ -field line, then  $E_T \cdot dl = |E_T|dl$ , giving:

$$V = \int_a^b |E_T| dl \tag{9.1.7}$$

Similarly, the current  $I$  can be obtained by the integral of  $H_T \cdot dl$  along any closed path around conductor (a). If that path is chosen to be an  $H$ -field line, such as the periphery  $C_a$  of the conductor, we will obtain:

$$I = \oint_{C_a} |H_T| dl \tag{9.1.8}$$

The surface charge accumulated on an infinitesimal area  $dl dz$  of conductor (a) is  $dQ = \rho_s dl dz$ , where  $\rho_s$  is the surface charge density. Because the conductors are assumed to be perfect, the boundary conditions require that  $\rho_s$  be equal to the normal component of the  $D$ -field, that is,  $\rho_s = \epsilon |E_T|$ . Thus,  $dQ = \epsilon |E_T| dl dz$ .

If we integrate over the periphery  $C_a$  of conductor (a), we will obtain the total surface charge per unit  $z$ -length:

$$Q' = \frac{dQ}{dz} = \oint_{C_a} \epsilon |E_T| dl$$

But because of the relationship  $|E_T| = \eta |H_T|$ , which follows from the first of Eqs. (9.1.1), we have:

$$Q' = \oint_{C_a} \epsilon |E_T| dl = \epsilon \eta \oint_{C_a} |H_T| dl = \epsilon \eta I \tag{9.1.9}$$

where we used Eq. (9.1.8). Because  $Q'$  is related to the capacitance per unit length and the voltage by  $Q' = C'V$ , we obtain

$$\boxed{Q' = C'V = \epsilon \eta I} \Rightarrow C' = \epsilon \eta \frac{I}{V} = \epsilon \frac{\eta}{Z}$$

Next, we consider an  $E$ -field line between points  $A$  and  $B$  on the two conductors. The magnetic flux through the infinitesimal area  $dl dz$  will be  $d\Phi = |B_T| dl dz = \mu |H_T| dl dz$  because the vector  $H_T$  is perpendicular to the area.

If we integrate from (a) to (b), we will obtain the total magnetic flux linking the two conductors per unit  $z$ -length:

$$\Phi' = \frac{d\Phi}{dz} = \int_a^b \mu |H_T| dl$$

replacing  $|H_T| = |E_T|/\eta$  and using Eq. (9.1.7), we find:

$$\Phi' = \int_a^b \mu |H_T| dl = \frac{\mu}{\eta} \int_a^b |E_T| dl = \frac{\mu}{\eta} V$$

The magnetic flux is related to the inductance via  $\Phi' = L'I$ . Therefore, we get:

$$\boxed{\Phi' = L'I = \frac{\mu}{\eta} V} \Rightarrow L' = \frac{\mu}{\eta} \frac{V}{I} = \mu \frac{Z}{\eta}$$

### Transmitted Power

The relationships among  $Z, L', C'$  can also be derived using energy considerations. The power transmitted along the line is obtained by integrating the  $z$ -component of the Poynting vector over the cross-section  $S$  of the line. For TEM modes we have  $P_z = |E_T|^2/2\eta$ , therefore,

$$P_T = \frac{1}{2\eta} \iint_S |E_T|^2 dx dy = \frac{1}{2\eta} \iint_S |\nabla_T \varphi|^2 dx dy \tag{9.1.10}$$

It can be shown in general that Eq. (9.1.10) can be rewritten as:

$$\boxed{P_T = \frac{1}{2} \text{Re}(V^* I) = \frac{1}{2} Z |I|^2 = \frac{1}{2Z} |V|^2} \tag{9.1.11}$$

We will verify this in the various examples below. It can be proved using the following Green's identity:

$$|\nabla_T \varphi|^2 + \varphi^* \nabla_T^2 \varphi = \nabla_T \cdot (\varphi^* \nabla_T \varphi)$$

Writing  $E_T = -\nabla_T \varphi$  and noting that  $\nabla_T^2 \varphi = 0$ , we obtain:

$$|E_T|^2 = -\nabla_T \cdot (\varphi^* E_T)$$

Then, the two-dimensional Gauss' theorem implies:

$$\begin{aligned}
P_T &= \frac{1}{2\eta} \iint_S |E_T|^2 dx dy = -\frac{1}{2\eta} \iint_S \nabla_T \cdot (\varphi^* E_T) dx dy \\
&= -\frac{1}{2\eta} \oint_{C_a} \varphi^* E_T \cdot (-\hat{\mathbf{n}}) dl - \frac{1}{2\eta} \oint_{C_b} \varphi^* E_T \cdot (-\hat{\mathbf{n}}) dl \\
&= \frac{1}{2\eta} \oint_{C_a} \varphi^* (E_T \cdot \hat{\mathbf{n}}) dl + \frac{1}{2\eta} \oint_{C_b} \varphi^* (E_T \cdot \hat{\mathbf{n}}) dl
\end{aligned}$$

where  $\hat{\mathbf{n}}$  are the outward normals to the conductors (the quantity  $-\hat{\mathbf{n}}$  is the normal outward from the region  $S$ .) Because the conductors are equipotential surfaces, we have  $\varphi^* = \varphi_a^*$  on conductor (a) and  $\varphi^* = \varphi_b^*$  on conductor (b). Using Eq. (9.1.9) and noting that  $E_T \cdot \hat{\mathbf{n}} = \pm |E_T|$  on conductors (a) and (b), we obtain:

$$\begin{aligned}
P_T &= \frac{1}{2\eta} \varphi_a^* \oint_{C_a} |E_T| dl - \frac{1}{2\eta} \varphi_b^* \oint_{C_b} |E_T| dl = \frac{1}{2\eta} \varphi_a^* \frac{Q'}{\epsilon} - \frac{1}{2\eta} \varphi_b^* \frac{Q'}{\epsilon} \\
&= \frac{1}{2} (\varphi_a^* - \varphi_b^*) \frac{Q'}{\epsilon \eta} = \frac{1}{2} V^* I = \frac{1}{2} Z |I|^2
\end{aligned}$$

The distribution of electromagnetic energy along the line is described by the time-averaged electric and magnetic energy densities per unit length, which are given by:

$$W'_e = \frac{1}{4} \epsilon \iint_S |E_T|^2 dx dy, \quad W'_m = \frac{1}{4} \mu \iint_S |H_T|^2 dx dy$$

Using Eq. (9.1.10), we may rewrite:

$$W'_e = \frac{1}{2} \epsilon \eta P_T = \frac{1}{2c} P_T, \quad W'_m = \frac{1}{2} \frac{\mu}{\eta} P_T = \frac{1}{2c} P_T$$

Thus,  $W'_e = W'_m$  and the total energy density is  $W' = W'_e + W'_m = P_T/c$ , which implies that the energy velocity will be  $v_{\text{en}} = P_T/W' = c$ . We may also express the energy densities in terms of the capacitance and inductance of the line:

$$W'_e = \frac{1}{4} C' |V|^2, \quad W'_m = \frac{1}{4} L' |I|^2 \quad (9.1.12)$$

### Power Losses, Resistance, and Conductance

Transmission line losses can be handled in the manner discussed in Sec. 8.2. The field patterns and characteristic impedance are determined assuming the conductors are perfectly conducting. Then, the losses due to the ohmic heating of the dielectric and the conductors can be calculated by Eqs. (8.2.5) and (8.2.9).

These losses can be quantified by two more characteristic parameters of the line, the resistance and conductance per unit length,  $R'$  and  $G'$ . The attenuation coefficients due to conductor and dielectric losses are then expressible in terms  $R'$ ,  $G'$  and  $Z$  by:

$$\alpha_c = \frac{R'}{2Z}, \quad \alpha_d = \frac{1}{2} G' Z \quad (9.1.13)$$

They can be derived in general terms as follows. The induced surface currents on the conductor walls are  $\mathbf{J}_s = \hat{\mathbf{n}} \times \mathbf{H}_T = \hat{\mathbf{n}} \times (\hat{\mathbf{z}} \times E_T)/\eta$ , where  $\hat{\mathbf{n}}$  is the outward normal to the wall.

Using the BAC-CAB rule, we find  $\mathbf{J}_s = \hat{\mathbf{z}}(\hat{\mathbf{n}} \cdot E_T)/\eta$ . But,  $\hat{\mathbf{n}}$  is parallel to  $E_T$  on the surface of conductor (a), and anti parallel on (b). Therefore,  $\hat{\mathbf{n}} \cdot E_T = \pm |E_T|$ . It follows that  $\mathbf{J}_s = \pm \hat{\mathbf{z}} |E_T|/\eta = \pm \hat{\mathbf{z}} |H_T|$ , pointing in the  $+z$  direction on (a) and  $-z$  direction on (b). Inserting these expressions into Eq. (8.2.8), we find for the conductor power loss per unit  $z$ -length:

$$P'_{\text{loss}} = \frac{dP_{\text{loss}}}{dz} = \frac{1}{2} R_s \oint_{C_a} |H_T|^2 dl + \frac{1}{2} R_s \oint_{C_b} |H_T|^2 dl \quad (9.1.14)$$

Because  $H_T$  is related to the total current  $I$  via Eq. (9.1.8), we may define the resistance per unit length  $R'$  through the relationship:

$$P'_{\text{loss}} = \frac{1}{2} R' |I|^2 \quad (\text{conductor ohmic losses}) \quad (9.1.15)$$

Using Eq. (9.1.11), we find for the attenuation coefficient:

$$\alpha_c = \frac{P'_{\text{loss}}}{2P_T} = \frac{\frac{1}{2} R' |I|^2}{2 \frac{1}{2} Z |I|^2} = \frac{R'}{2Z} \quad (9.1.16)$$

If the dielectric between the conductors is slightly conducting with conductivity  $\sigma_d$  or loss tangent  $\tan \delta = \sigma_d/\epsilon\omega$ , then there will be some current flow between the two conductors.

The induced shunt current per unit  $z$ -length is related to the conductance by  $I'_d = G'V$ . The shunt current density within the dielectric is  $\mathbf{J}_d = \sigma_d E_T$ . The total shunt current flowing out of conductor (a) towards conductor (b) is obtained by integrating  $\mathbf{J}_d$  around the periphery of conductor (a):

$$I'_d = \oint_{C_a} \mathbf{J}_d \cdot \hat{\mathbf{n}} dl = \sigma_d \oint_{C_a} |E_T| dl$$

Using Eq. (9.1.9), we find:

$$I'_d = \sigma_d \frac{Q'}{\epsilon} = G'V \Rightarrow G' = \frac{\sigma_d}{\epsilon} C' = \sigma_d \frac{\eta}{Z}$$

It follows that the dielectric loss constant (8.2.5) will be:

$$\alpha_d = \frac{1}{2} \sigma_d \eta = \frac{1}{2} G' Z$$

Alternatively, the power loss per unit length due to the shunt current will be  $P'_d = \text{Re}(I'_d V^*)/2 = G' |V|^2/2$ , and therefore,  $\alpha_d$  can be computed from:

$$\alpha_d = \frac{P'_d}{2P_T} = \frac{\frac{1}{2} G' |V|^2}{2 \frac{1}{2} |V|^2} = \frac{1}{2} G' Z$$

It is common practice to express the dielectric losses and shunt conductance in terms of the loss tangent  $\tan \delta$  and the wavenumber  $\beta = \omega/c = \omega\epsilon\eta$ :

$$\alpha_d = \frac{1}{2}\sigma_d\eta = \frac{1}{2}\omega\epsilon\eta \tan \delta = \frac{1}{2}\beta \tan \delta \quad \text{and} \quad G' = \frac{\sigma_d}{\epsilon}C' = \omega C' \tan \delta \quad (9.1.17)$$

Next, we discuss four examples: the parallel plate line, the microstrip line, the coaxial cable, and the two-wire line. In each case, we discuss the nature of the electrostatic problem and determine the characteristic impedance  $Z$  and the attenuation coefficients  $\alpha_c$  and  $\alpha_d$ .

### 9.2 Parallel Plate Lines

The parallel plate line shown in Fig. 9.2.1 consists of two parallel conducting plates of width  $w$  separated by height  $h$  by a dielectric material  $\epsilon$ . Examples of such lines are microstrip lines used in microwave integrated circuits.

For arbitrary values of  $w$  and  $h$ , the fringing effects at the ends of the plates cannot be ignored. In fact, fringing requires the fields to have longitudinal components, and therefore TEM modes are not strictly-speaking supported.

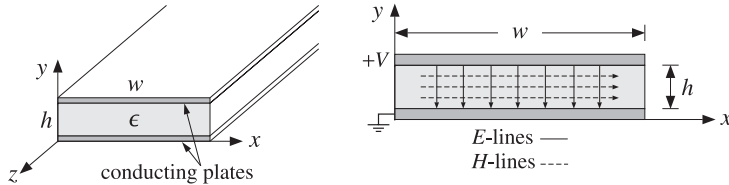


Fig. 9.2.1 Parallel plate transmission line.

However, assuming the width is much larger than the height,  $w \gg h$ , we may ignore the fringing effects and assume that the fields have no dependence on the  $x$ -coordinate.

The electrostatic problem is equivalent to that of a parallel plate capacitor. Thus, the electric field will have only a  $y$  component and will be constant between the plates. Similarly, the magnetic field will have only an  $x$  component. It follows from Eqs. (9.1.7) and (9.1.8) that:

$$V = -E_y h, \quad I = H_x w$$

Therefore, the characteristic impedance of the line will be:

$$Z = \frac{V}{I} = \frac{-E_y h}{H_x w} = \eta \frac{h}{w} \quad (9.2.1)$$

where we used  $E_y = -\eta H_x$ . The transmitted power is obtained from Eq. (9.1.10):

$$P_T = \frac{1}{2\eta} |E_y|^2 (wh) = \frac{1}{2\eta} \frac{V^2}{h^2} wh = \frac{1}{2\eta} \frac{w}{h} V^2 = \frac{1}{2Z} V^2 = \frac{1}{2} Z I^2 \quad (9.2.2)$$

The inductance and capacitance per unit length are obtained from Eq. (9.1.5):

$$L' = \mu \frac{h}{w}, \quad C' = \epsilon \frac{w}{h} \quad (9.2.3)$$

The surface current on the top conductor is  $\mathbf{J}_s = \hat{\mathbf{n}} \times \mathbf{H} = (-\hat{\mathbf{y}}) \times \mathbf{H} = \hat{\mathbf{z}} H_x$ . On the bottom conductor, it will be  $\mathbf{J}_s = -\hat{\mathbf{z}} H_x$ . Therefore, the power loss per unit  $z$ -length is obtained from Eq. (8.2.8):

$$P'_{\text{loss}} = 2 \frac{1}{2} R_s |H_x|^2 w = \frac{1}{w} R_s I^2$$

Comparing with Eq. (9.1.15), we identify the resistance per unit length  $R' = 2R_s/w$ . Then, the attenuation constant due to conductor losses will be:

$$\alpha_c = \frac{P'_{\text{loss}}}{2P_T} = \frac{R'}{2Z} = \frac{R_s}{wZ} = \frac{R_s}{h\eta} \quad (9.2.4)$$

### 9.3 Microstrip Lines

Practical microstrip lines, shown in Fig. 9.3.1, have width-to-height ratios  $w/h$  that are not necessarily much greater than unity, and can vary over the interval  $0.1 < w/h < 10$ . Typical heights  $h$  are of the order of millimeters.

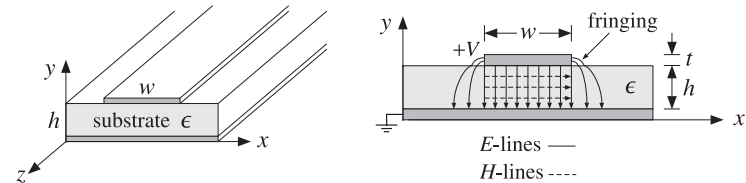


Fig. 9.3.1 A microstrip transmission line.

Fringing effects cannot be ignored completely and the simple assumptions about the fields of the parallel plate line are not valid. For example, assuming a propagating wave in the  $z$ -direction with  $z, t$  dependence of  $e^{j\omega t - j\beta z}$  with a common  $\beta$  in the dielectric and air, the longitudinal-transverse decomposition (8.1.5) gives:

$$\nabla_T E_z \times \hat{\mathbf{z}} - j\beta \hat{\mathbf{z}} \times \mathbf{E}_T = -j\omega \mu \mathbf{H}_T \quad \Rightarrow \quad \hat{\mathbf{z}} \times (\nabla_T E_z + j\beta \mathbf{E}_T) = j\omega \mu \mathbf{H}_T$$

In particular, we have for the  $x$ -component:

$$\partial_y E_z + j\beta E_y = -j\omega \mu H_x$$

The boundary conditions require that the components  $H_x$  and  $D_y = \epsilon E_y$  be continuous across the dielectric-air interface. This gives the interface conditions:

$$\begin{aligned} \partial_y E_z^{\text{air}} + j\beta E_y^{\text{air}} &= \partial_y E_z^{\text{diel}} + j\beta E_y^{\text{diel}} \\ \epsilon_0 E_y^{\text{air}} &= \epsilon E_y^{\text{diel}} \end{aligned}$$

Combining the two conditions, we obtain:

$$\partial_y (E_z^{\text{diel}} - E_z^{\text{air}}) = j\beta \frac{\epsilon - \epsilon_0}{\epsilon} E_y^{\text{air}} = j\beta \frac{\epsilon - \epsilon_0}{\epsilon_0} E_y^{\text{diel}} \quad (9.3.1)$$

Because  $E_y$  is non-zero on either side of the interface, it follows that the left-hand side of Eq. (9.3.1) cannot be zero and the wave cannot be assumed to be strictly TEM.

However,  $E_y$  is small in both the air and the dielectric in the fringing regions (to the left and right of the upper conductor). This gives rise to the so-called *quasi-TEM approximation* in which the fields are assumed to be approximately TEM and the effect of the deviation from TEM is taken into account by empirical formulas for the line impedance and velocity factor.

In particular, the air-dielectric interface is replaced by an *effective* dielectric, filling uniformly the entire space, and in which there would be a TEM propagating mode. If we denote by  $\epsilon_{\text{eff}}$  the relative permittivity of the effective dielectric, the wavelength and velocity factor of the line will be given in terms of their free-space values  $\lambda_0, c_0$ :

$$\lambda = \frac{\lambda_0}{\sqrt{\epsilon_{\text{eff}}}}, \quad c = \frac{c_0}{\sqrt{\epsilon_{\text{eff}}}} \quad (9.3.2)$$

There exist many empirical formulas for the characteristic impedance of the line and the effective dielectric constant. Hammerstad and Jensen's are some of the most accurate ones [469,475]:

$$\epsilon_{\text{eff}} = \frac{\epsilon_r + 1}{2} + \frac{\epsilon_r - 1}{2} \left(1 + \frac{10}{u}\right)^{-ab}, \quad u = \frac{w}{h} \quad (9.3.3)$$

where  $\epsilon_r = \epsilon/\epsilon_0$  is the relative permittivity of the dielectric and the quantities  $a, b$  are defined by:

$$a = 1 + \frac{1}{49} \ln \left[ \frac{u^4 + (u/52)^2}{u^4 + 0.432} \right] + \frac{1}{18.7} \ln \left[ 1 + \left( \frac{u}{18.1} \right)^3 \right] \quad (9.3.4)$$

$$b = 0.564 \left( \frac{\epsilon_r - 0.9}{\epsilon_r + 3} \right)^{0.053}$$

The accuracy of these formulas is better than 0.01% for  $u < 1$  and 0.03% for  $u < 1000$ . Similarly, the characteristic impedance is given by the empirical formula:

$$Z = \frac{\eta_0}{2\pi\sqrt{\epsilon_{\text{eff}}}} \ln \left[ \frac{f(u)}{u} + \sqrt{1 + \frac{4}{u^2}} \right] \quad (9.3.5)$$

where  $\eta_0 = \sqrt{\mu_0/\epsilon_0}$  and the function  $f(u)$  is defined by:

$$f(u) = 6 + (2\pi - 6) \exp \left[ - \left( \frac{30.666}{u} \right)^{0.7528} \right] \quad (9.3.6)$$

The accuracy is better than 0.2% for  $0.1 \leq u \leq 100$  and  $\epsilon_r < 128$ . In the limit of large ratio  $w/h$ , or,  $u \rightarrow \infty$ , Eqs. (9.3.3) and (9.3.5) tend to those of the parallel plate line of the previous section:

$$\epsilon_{\text{eff}} \rightarrow \epsilon_r, \quad Z \rightarrow \frac{\eta_0}{\sqrt{\epsilon_r}} \frac{h}{w} = \eta \frac{h}{w}$$

Some typical substrate dielectric materials used in microstrip lines are alumina, a ceramic form of  $\text{Al}_2\text{O}_3$  with  $\epsilon_r = 9.8$ , and RT-Duroid, a teflon composite material with  $\epsilon_r = 2.2$ . Practical values of the width-to-height ratio are in the range  $0.1 \leq u \leq 10$  and practical values of characteristic impedances are between 10–200 ohm. Fig. 9.3.2 shows the dependence of  $Z$  and  $\epsilon_{\text{eff}}$  on  $u$  for the two cases of  $\epsilon_r = 2.2$  and  $\epsilon_r = 9.8$ .

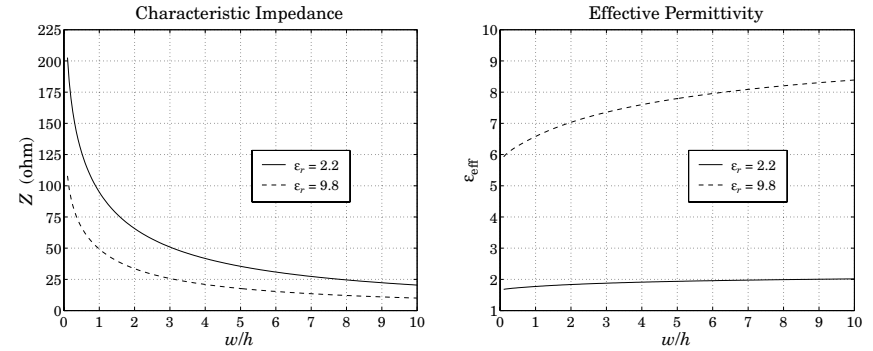


Fig. 9.3.2 Characteristic impedance and effective permittivity of microstrip line.

The synthesis of a microstrip line requires that we determine the ratio  $w/h$  that will achieve a given characteristic impedance  $Z$ . The inverse of Eq. (9.3.5)—solving for  $u$  in terms of  $Z$ —is not practical. Direct synthesis empirical equations exist [470,475], but are not as accurate as (9.3.5). Given a desired  $Z$ , the ratio  $u = w/h$  is calculated as follows. If  $u \leq 2$ ,

$$u = \frac{8}{e^A - 2e^{-A}} \quad (9.3.7)$$

and, if  $u > 2$ ,

$$u = \frac{\epsilon_r - 1}{\pi\epsilon_r} \left[ \ln(B - 1) + 0.39 - \frac{0.61}{\epsilon_r} \right] + \frac{2}{\pi} [B - 1 - \ln(2B - 1)] \quad (9.3.8)$$

where  $A, B$  are given by:

$$A = \pi\sqrt{2(\epsilon_r + 1)} \frac{Z}{\eta_0} + \frac{\epsilon_r - 1}{\epsilon_r + 1} \left( 0.23 + \frac{0.11}{\epsilon_r} \right) \quad (9.3.9)$$

$$B = \frac{\pi}{2\sqrt{\epsilon_r}} \frac{\eta_0}{Z}$$

The accuracy of these formulas is about 1%. The method can be improved iteratively by a process of refinement to achieve essentially the same accuracy as Eq. (9.3.5). Starting with  $u$  computed from Eqs. (9.3.7) and (9.3.8), a value of  $Z$  is computed through Eq. (9.3.5). If that  $Z$  is more than, say, 0.2% off from the desired value of the line

impedance, then  $u$  is slightly changed, and so on, until the desired level of accuracy is reached [475]. Because  $Z$  is monotonically decreasing with  $u$ , if  $Z$  is less than the desired value, then  $u$  is decreased by a small percentage, else,  $u$  is increased by the same percentage.

The three MATLAB functions `mstripa`, `mstrips`, and `mstrip` implement the analysis, synthesis, and refinement procedures. They have usage:

```
[eff,Z] = mstripa(er,u);           % analysis equations (9.3.3) and (9.3.5)
u = mstrips(er,Z);               % synthesis equations (9.3.7) and (9.3.8)
[u,N] = mstrip(er,Z,per);        % refinement
```

The function `mstripa` accepts also a vector of several  $u$ 's, returning the corresponding vector of values of  $\epsilon_{\text{eff}}$  and  $Z$ . In `mstrip`, the output  $N$  is the number of iterations required for convergence, and `per` is the desired percentage error, which defaults to 0.2% if this parameter is omitted.

**Example 9.3.1:** Given  $\epsilon_r = 2.2$  and  $u = w/h = 2, 4, 6$ , the effective permittivities and impedances are computed from the MATLAB call:

```
u = [2; 4; 6];
[eff, Z] = mstripa(er,u);
```

The resulting output vectors are:

$$u = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \Rightarrow \epsilon_{\text{eff}} = \begin{bmatrix} 1.8347 \\ 1.9111 \\ 1.9585 \end{bmatrix}, \quad Z = \begin{bmatrix} 65.7273 \\ 41.7537 \\ 30.8728 \end{bmatrix} \text{ ohm}$$

**Example 9.3.2:** To compare the outputs of `mstrips` and `mstrip`, we design a microstrip line with  $\epsilon_r = 2.2$  and characteristic impedance  $Z = 50$  ohm. We find:

$$u = \text{mstrips}(2.2, 50) = 3.0779 \Rightarrow [\epsilon_{\text{eff}}, Z] = \text{mstripa}(2.2, u) = [1.8811, 50.0534]$$

$$u = \text{mstrip}(2.2, 50) = 3.0829 \Rightarrow [\epsilon_{\text{eff}}, Z] = \text{mstripa}(2.2, u) = [1.8813, 49.9990]$$

The first solution has an error of 0.107% from the desired 50 ohm impedance, and the second, a 0.002% error.

As another example, if  $Z = 100 \Omega$ , the function `mstrips` results in  $u = 0.8949$ ,  $Z = 99.9495 \Omega$ , and a 0.050% error, whereas `mstrip` gives  $u = 0.8939$ ,  $Z = 99.9980 \Omega$ , and a 0.002% error.  $\square$

In using microstrip lines several other effects must be considered, such as finite strip thickness, frequency dispersion, dielectric and conductor losses, radiation, and surface waves. Guidelines for such effects can be found in [469-475].

The dielectric losses are obtained from Eq. (9.1.17) by multiplying it by an effective dielectric *filling* factor  $q$ :

$$\alpha_d = q \frac{\omega}{2c} \tan \delta = \frac{f}{c_0} \pi q \sqrt{\epsilon_{\text{eff}}} \tan \delta = \frac{1}{\lambda_0} \pi q \sqrt{\epsilon_{\text{eff}}} \tan \delta, \quad q = \frac{1 - \epsilon_{\text{eff}}^{-1}}{1 - \epsilon_r^{-1}} \quad (9.3.10)$$

Typical values of the loss tangent are of the order of 0.001 for alumina and duroid substrates. The conductor losses are approximately computed from Eq. (9.2.4):

$$\alpha_c = \frac{R_s}{wZ} \quad (9.3.11)$$

## 9.4 Coaxial Lines

The coaxial cable, depicted in Fig. 9.4.1, is the most widely used TEM transmission line. It consists of two concentric conductors of inner and outer radii of  $a$  and  $b$ , with the space between them filled with a dielectric  $\epsilon$ , such as polyethylene or teflon.

The equivalent electrostatic problem can be solved conveniently in cylindrical coordinates  $\rho, \phi$ . The potential  $\varphi(\rho, \phi)$  satisfies Laplace's equation:

$$\nabla_T^2 \varphi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \varphi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \varphi}{\partial \phi^2} = 0$$

Because of the cylindrical symmetry, the potential does not depend on the azimuthal angle  $\phi$ . Therefore,

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \varphi}{\partial \rho} \right) = 0 \Rightarrow \rho \frac{\partial \varphi}{\partial \rho} = B \Rightarrow \varphi(\rho) = A + B \ln \rho$$

where  $A, B$  are constants of integration. Assuming the outer conductor is grounded,  $\varphi(\rho) = 0$  at  $\rho = b$ , and the inner conductor is held at voltage  $V$ ,  $\varphi(a) = V$ , the constants  $A, B$  are determined to be  $B = -V \ln(b/a)$  and  $A = -B \ln b$ , resulting in the potential:

$$\varphi(\rho) = \frac{V}{\ln(b/a)} \ln(b/\rho) \quad (9.4.1)$$

It follows that the electric field will have only a radial component,  $E_\rho = -\partial_\rho \varphi$ , and the magnetic field only an azimuthal component  $H_\phi = E_\rho / \eta$ :

$$E_\rho = \frac{V}{\ln(b/a)} \frac{1}{\rho}, \quad H_\phi = \frac{V}{\eta \ln(b/a)} \frac{1}{\rho} \quad (9.4.2)$$

Integrating  $H_\phi$  around the inner conductor we obtain the current:

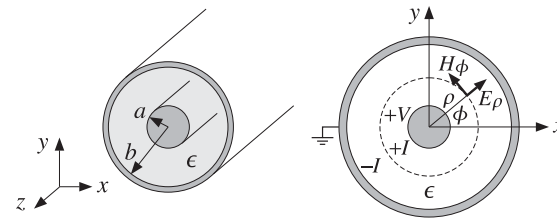


Fig. 9.4.1 Coaxial transmission line.

$$I = \int_0^{2\pi} H_\phi \rho d\phi = \int_0^{2\pi} \frac{V}{\eta \ln(b/a)} \frac{1}{\rho} \rho d\phi = \frac{2\pi V}{\eta \ln(b/a)} \quad (9.4.3)$$

It follows that the characteristic impedance of the line  $Z = V/I$ , and hence the inductance and capacitance per unit length, will be:

$$Z = \frac{\eta}{2\pi} \ln(b/a), \quad L' = \frac{\mu}{2\pi} \ln(b/a), \quad C' = \frac{2\pi\epsilon}{\ln(b/a)} \quad (9.4.4)$$

Using Eq. (9.4.3) into (9.4.2), we may express the magnetic field in the form:

$$H_\phi = \frac{I}{2\pi\rho} \quad (9.4.5)$$

This is also obtainable by the direct application of Ampère's law around the loop of radius  $\rho$  encircling the inner conductor, that is,  $I = (2\pi\rho)H_\phi$ .

The transmitted power can be expressed either in terms of the voltage  $V$  or in terms of the maximum value of the electric field inside the line, which occurs at  $\rho = a$ , that is,  $E_a = V/(a \ln(b/a))$ :

$$P_T = \frac{1}{2Z} |V|^2 = \frac{\pi |V|^2}{\eta \ln(b/a)} = \frac{1}{\eta} |E_a|^2 (\pi a^2) \ln(b/a) \quad (9.4.6)$$

**Example 9.4.1:** A commercially available polyethylene-filled RG-58/U cable is quoted to have impedance of 53.5  $\Omega$ , velocity factor of 66 percent, inner conductor radius  $a = 0.406$  mm (AWG 20-gauge wire), and maximum operating RMS voltage of 1900 volts. Determine the outer-conductor radius  $b$ , the capacitance per unit length  $C'$ , the maximum power  $P_T$  that can be transmitted, and the maximum electric field inside the cable. What should be the outer radius  $b$  if the impedance were required to be exactly 50  $\Omega$ ?

**Solution:** Polyethylene has a relative dielectric constant of  $\epsilon_r = 2.25$ , so that  $n = \sqrt{\epsilon_r} = 1.5$ . The velocity factor is  $c/c_0 = 1/n = 0.667$ . Given that  $\eta = \eta_0/n = 376.73/1.5 = 251.15$   $\Omega$ , we have:

$$Z = \frac{\eta}{2\pi} \ln(b/a) \Rightarrow b = ae^{2\pi Z/\eta} = 0.406e^{2\pi \cdot 53.5/251.15} = 1.548 \text{ mm}$$

Therefore,  $b/a = 3.81$ . If  $Z = 50$ , the above calculation would give  $b = 1.418$  mm and  $b/a = 3.49$ . The capacitance per unit length is found from:

$$C' = \epsilon \frac{\eta}{Z} = \frac{1}{cZ} = \frac{n}{c_0 Z} = \frac{1.5}{3 \times 10^8 \times 53.5} = 93.46 \text{ pF/m}$$

For  $Z = 50$   $\Omega$ , we find  $C' = 100$  pF/m. The peak voltage is related to its RMS value by  $|V| = \sqrt{2}V_{\text{rms}}$ . It follows that the maximum power transmitted is:

$$P_T = \frac{1}{2Z} |V|^2 = \frac{V_{\text{rms}}^2}{Z} = \frac{1900^2}{53.5} = 67.5 \text{ kW}$$

The peak value of the electric field occurring at the inner conductor will be:

$$|E_a| = \frac{|V|}{a \ln(b/a)} = \frac{\sqrt{2}V_{\text{rms}}}{a \ln(b/a)} = \frac{\sqrt{2}1900}{0.406 \times 10^{-2} \ln(3.096/0.406)} = 0.5 \text{ MV/m}$$

This is to be compared with the dielectric breakdown of air of 3 MV/m. For a 73- $\Omega$  RG-59/U cable with  $a = 0.322$  mm (AWG 22-gauge wire), we find  $b = 2$  mm,  $C' = 68.5$  pF/m,  $P_T = 49.5$  kW, and  $E_{\text{max}} = 0.46$  MV/m.  $\square$

**Example 9.4.2:** Most cables have a nominal impedance of either 50 or 75  $\Omega$ . The precise value depends on the manufacturer and the cable. For example, a 50- $\Omega$  cable might actually have an impedance of 52  $\Omega$  and a 75- $\Omega$  cable might actually be a 73- $\Omega$  cable.

The table below lists some commonly used cables with their AWG-gauge number of the inner conductor, the inner conductor radius  $a$  in mm, and their nominal impedance. Their dielectric filling is polyethylene with  $\epsilon_r = 2.25$  or  $n = \sqrt{\epsilon_r} = 1.5$ .

type	AWG	$a$	$Z$
RG-6/U	18	0.512	75
RG-8/U	11	1.150	50
RG-11/U	14	0.815	75
RG-58/U	20	0.406	50
RG-59/U	22	0.322	75
RG-174/U	26	0.203	50
RG-213/U	13	0.915	50

The most commonly used cables are 50- $\Omega$  ones, such as the RG-58/U. Home cable-TV uses 75- $\Omega$  cables, such as the RG-59/U or RG-6/U.

The thin ethernet computer network, known as 10base-2, uses RG-58/U or RG-58A/U, which is similar to the RG-58/U but has a stranded inner copper core. Thick ethernet (10base-5) uses the thicker RG-8/U cable.

Because a dipole antenna has an input impedance of about 73  $\Omega$ , the RG-11, RG-6, and RG-59 75- $\Omega$  cables can be used to feed the antenna.  $\square$

Next, we determine the attenuation coefficient due to conductor losses. The power loss per unit length is given by Eq. (9.1.14). The magnetic fields at the surfaces of conductors (a) and (b) are obtained from Eq. (9.4.5) by setting  $\rho = a$  and  $\rho = b$ :

$$H_a = \frac{I}{2\pi a}, \quad H_b = \frac{I}{2\pi b}$$

Because these are independent of the azimuthal angle, the integrations around the peripheries  $dl = ad\phi$  or  $dl = bd\phi$  will contribute a factor of  $(2\pi a)$  or  $(2\pi b)$ . Thus,

$$P'_{\text{loss}} = \frac{1}{2} R_s [(2\pi a)|H_a|^2 + (2\pi b)|H_b|^2] = \frac{R_s |I|^2}{4\pi} \left( \frac{1}{a} + \frac{1}{b} \right) \quad (9.4.7)$$

It follows that:

$$\alpha_c = \frac{P'_{\text{loss}}}{2P_T} = \frac{R_s |I|^2 \left( \frac{1}{a} + \frac{1}{b} \right)}{2 \frac{1}{2} Z |I|^2}$$

Using Eq. (9.4.4), we finally obtain:

$$\alpha_c = \frac{R_s}{2\eta} \frac{\left(\frac{1}{a} + \frac{1}{b}\right)}{\ln\left(\frac{b}{a}\right)} \quad (9.4.8)$$

The ohmic losses in the dielectric are described by Eq. (9.1.17). The total attenuation constant will be the sum of the conductor and dielectric attenuations:

$$\alpha = \alpha_c + \alpha_d = \frac{R_s}{2\eta} \frac{\left(\frac{1}{a} + \frac{1}{b}\right)}{\ln\left(\frac{b}{a}\right)} + \frac{\omega}{2c} \tan \delta \quad (\text{attenuation}) \quad (9.4.9)$$

The attenuation in dB/m will be  $\alpha_{dB} = 8.686 \alpha$ . This expression tends to somewhat underestimate the actual losses, but it is generally a good approximation. The  $\alpha_c$  term grows in frequency like  $\sqrt{f}$  and the term  $\alpha_d$ , like  $f$ .

The smaller the dimensions  $a, b$ , the larger the attenuation. The loss tangent  $\tan \delta$  of a typical polyethylene or teflon dielectric is of the order of 0.0004-0.0009 up to about 3 GHz.

The ohmic losses and the resulting heating of the dielectric and conductors also limits the power rating of the line. For example, if the maximum supported voltage is 1900 volts as in Example 9.4.2, the RMS value of the current for an RG-58/U line would be  $I_{rms} = 1900/53.5 = 35.5$  amps, which would melt the conductors. Thus, the actual power rating is much smaller than that suggested by the maximum voltage rating. The typical power rating of an RG-58/U cable is 1 kW, 200 W, and 80 W at 10 MHz, 200 MHz, and 1 GHz.

**Example 9.4.3:** The table below lists the nominal attenuations in dB per 100 feet of the RG-8/U and RG-213/U cables. The data are from [843].

$f$ (MHz)	50	100	200	400	900	1000	3000	5000
$\alpha$ (dB/100ft)	1.3	1.9	2.7	4.1	7.5	8.0	16.0	27.0

Both are 50-ohm cables and their radii  $a$  are 1.15 mm and 0.915 mm for RG-8/U and RG-213/U. In order to compare these ratings with Eq. (9.4.9), we took  $a$  to be the average of these two values, that is,  $a = 1.03$  mm. The required value of  $b$  to give a 50-ohm impedance is  $b = 3.60$  mm.

Fig. 9.4.2 shows the attenuations calculated from Eq. (9.4.9) and the nominal ones from the table. We assumed copper conductors with  $\sigma = 5.8 \times 10^7$  S/m and polyethylene dielectric with  $n = 1.5$ , so that  $\eta = \eta_0/n = 376.73/1.5 = 251.15 \Omega$  and  $c = c_0/n = 2 \times 10^8$  m/sec. The loss tangent was taken to be  $\tan \delta = 0.0007$ .

The conductor and dielectric attenuations  $\alpha_c$  and  $\alpha_d$  become equal around 2.3 GHz, and  $\alpha_d$  dominates after that.

It is evident that the useful operation of the cable is restricted to frequencies up to 1 GHz. Beyond that, the attenuations are too excessive and the cable may be used only for short lengths.  $\square$

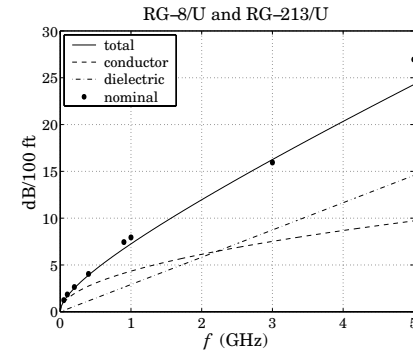


Fig. 9.4.2 Attenuation coefficient  $\alpha$  versus frequency.

### Optimum Coaxial Cables

Given a fixed outer-conductor radius  $b$ , one may ask three optimization questions: What is the optimum value of  $a$ , or equivalently, the ratio  $b/a$  that (a) minimizes the electric field  $E_a$  inside the guide, (b) maximizes the power transfer  $P_T$ , and (c) minimizes the conductor attenuation  $\alpha_c$ .

The three quantities  $E_a, P_T, \alpha_c$  can be thought of as functions of the ratio  $x = b/a$  and take the following forms:

$$E_a = \frac{V}{b} \frac{x}{\ln x}, \quad P_T = \frac{1}{\eta} |E_a|^2 \pi b^2 \frac{\ln x}{x^2}, \quad \alpha_c = \frac{R_s}{2\eta b} \frac{x+1}{\ln x} \quad (9.4.10)$$

Setting the derivatives of the three functions of  $x$  to zero, we obtain the three conditions: (a)  $\ln x = 1$ , (b)  $\ln x = 1/2$ , and (c)  $\ln x = 1 + 1/x$ , with solutions (a)  $b/a = e^1 = 2.7183$ , (b)  $b/a = e^{1/2} = 1.6487$  and (c)  $b/a = 3.5911$ .

Unfortunately, the three optimization problems have three different answers, and it is not possible to satisfy them simultaneously. The corresponding impedances  $Z$  for the three values of  $b/a$  are 60  $\Omega$ , 30  $\Omega$ , and 76.7  $\Omega$  for an air-filled line and 40  $\Omega$ , 20  $\Omega$ , and 51  $\Omega$  for a polyethylene-filled line.

The value of 50  $\Omega$  is considered to be a compromise between 30 and 76.7  $\Omega$  corresponding to maximum power and minimum attenuation. Actually, the minimum of  $\alpha_c$  is very broad and any neighboring value to  $b/a = 3.5911$  will result in an  $\alpha_c$  very near its minimum.

### Higher Modes

The TEM propagation mode is the dominant one and has no cutoff frequency. However, TE and TM modes with higher cutoff frequencies also exist in coaxial lines [444], with the lowest being a  $TE_{11}$  mode with cutoff wavelength and frequency:

$$\lambda_c = 1.873 \frac{\pi}{2} (a + b), \quad f_c = \frac{c}{\lambda_c} = \frac{c_0}{n\lambda_c} \quad (9.4.11)$$



This is usually approximated by  $\lambda_c = \pi(a + b)$ . Thus, the operation of the TEM mode is restricted to frequencies that are less than  $f_c$ .

**Example 9.4.4:** For the RG-58/U line of Example 9.4.2, we have  $a = 0.406$  mm and  $b = 1.548$  mm, resulting in  $\lambda_c = 1.873\pi(a + b)/2 = 5.749$  mm, which gives for the cutoff frequency  $f_c = 20/0.5749 = 34.79$  GHz, where we used  $c = c_0/n = 20$  GHz cm.

For the RG-8/U and RG-213/U cables, we may use  $a = 1.03$  mm and  $b = 3.60$  as in Example 9.4.3, resulting in  $\lambda_c = 13.622$  mm, and cutoff frequency of  $f_c = 14.68$  GHz.

The above cutoff frequencies are far above the useful operating range over which the attenuation of the line is acceptable.  $\square$

### 9.5 Two-Wire Lines

The two-wire transmission line consists of two parallel cylindrical conductors of radius  $a$  separated by distance  $d$  from each other, as shown in Fig. 9.5.1.

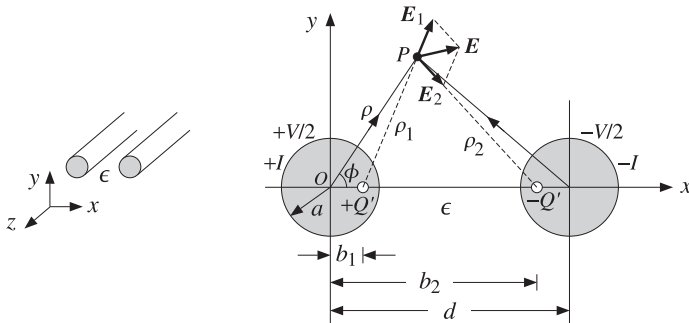


Fig. 9.5.1 Two-wire transmission line.

We assume that the conductors are held at potentials  $\pm V/2$  with charge per unit length  $\pm Q'$ . The electrostatic problem can be solved by the standard technique of replacing the finite-radius conductors by two thin line-charges  $\pm Q'$ .

The locations  $b_1$  and  $b_2$  of the line-charges are determined by the requirement that the cylindrical surfaces of the original conductors be equipotential surfaces, the idea being that if these equipotential surfaces were to be replaced by the conductors, the field patterns will not be disturbed.

The electrostatic problem of the two lines is solved by invoking superposition and adding the potentials due to the two lines, so that the potential at the field point  $P$  will be:

$$\varphi(\rho, \phi) = -\frac{Q'}{2\pi\epsilon} \ln \rho_1 - \frac{-Q'}{2\pi\epsilon} \ln \rho_2 = \frac{Q'}{2\pi\epsilon} \ln \left( \frac{\rho_2}{\rho_1} \right) \tag{9.5.1}$$

where the  $\rho_1, \rho_2$  are the distances from the line charges to  $P$ . From the triangles  $OP(+Q')$  and  $OP(-Q')$ , we may express these distances in terms of the polar coordinates  $\rho, \phi$  of the point  $P$ :

$$\rho_1 = \sqrt{\rho^2 - 2\rho b_1 \cos \phi + b_1^2}, \quad \rho_2 = \sqrt{\rho^2 - 2\rho b_2 \cos \phi + b_2^2} \tag{9.5.2}$$

Therefore, the potential function becomes:

$$\varphi(\rho, \phi) = \frac{Q'}{2\pi\epsilon} \ln \left( \frac{\rho_2}{\rho_1} \right) = \frac{Q'}{2\pi\epsilon} \ln \left( \sqrt{\frac{\rho^2 - 2\rho b_2 \cos \phi + b_2^2}{\rho^2 - 2\rho b_1 \cos \phi + b_1^2}} \right) \tag{9.5.3}$$

In order that the surface of the left conductor at  $\rho = a$  be an equipotential surface, that is,  $\varphi(a, \phi) = V/2$ , the ratio  $\rho_2/\rho_1$  must be a constant independent of  $\phi$ . Thus, we require that for some constant  $k$  and all angles  $\phi$ :

$$\left. \frac{\rho_2}{\rho_1} \right|_{\rho=a} = \sqrt{\frac{a^2 - 2ab_2 \cos \phi + b_2^2}{a^2 - 2ab_1 \cos \phi + b_1^2}} = k$$

which can be rewritten as:

$$a^2 - 2ab_2 \cos \phi + b_2^2 = k^2(a^2 - 2ab_1 \cos \phi + b_1^2)$$

This will be satisfied for all  $\phi$  provided we have:

$$a^2 + b_2^2 = k^2(a^2 + b_1^2), \quad b_2 = k^2 b_1$$

These may be solved for  $b_1, b_2$  in terms of  $k$ :

$$b_2 = ka, \quad b_1 = \frac{a}{k} \tag{9.5.4}$$

The quantity  $k$  can be expressed in terms of  $a, d$  by noting that because of symmetry, the charge  $-Q'$  is located also at distance  $b_1$  from the center of the right conductor. Therefore,  $b_1 + b_2 = d$ . This gives the condition:

$$b_1 + b_2 = d \Rightarrow a(k + k^{-1}) = d \Rightarrow k + k^{-1} = \frac{d}{a}$$

with solution for  $k$ :

$$k = \frac{d}{2a} + \sqrt{\left(\frac{d}{2a}\right)^2 - 1} \tag{9.5.5}$$

An alternative expression is obtained by setting  $k = e^\chi$ . Then, we have the condition:

$$b_1 + b_2 = d \Rightarrow a(e^\chi + e^{-\chi}) = 2a \cosh \chi = d \Rightarrow \chi = \operatorname{acosh} \left( \frac{d}{2a} \right) \tag{9.5.6}$$

Because  $\chi = \ln k$ , we obtain for the potential value of the left conductor:

$$\varphi(a, \phi) = \frac{Q'}{2\pi\epsilon} \ln k = \frac{Q'}{2\pi\epsilon} \chi = \frac{1}{2}V$$

This gives for the capacitance per unit length:

$$C' = \frac{Q'}{V} = \frac{\pi\epsilon}{\chi} = \frac{\pi\epsilon}{\operatorname{acosh}\left(\frac{d}{2a}\right)} \quad (9.5.7)$$

The corresponding line impedance and inductance are obtained from  $C' = \epsilon\eta/Z$  and  $L' = \mu Z/\eta$ . We find:

$$Z = \frac{\eta}{\pi}\chi = \frac{\eta}{\pi}\operatorname{acosh}\left(\frac{d}{2a}\right) \quad L' = \frac{\mu}{\pi}\chi = \frac{\mu}{\pi}\operatorname{acosh}\left(\frac{d}{2a}\right) \quad (9.5.8)$$

In the common case when  $d \gg a$ , we have approximately  $k \simeq d/a$ , and therefore,  $\chi = \ln k = \ln(d/a)$ . Then,  $Z$  can be written approximately as:

$$Z = \frac{\eta}{\pi} \ln(d/a) \quad (9.5.9)$$

To complete the electrostatic problem and determine the electric and magnetic fields of the TEM mode, we replace  $b_2 = ak$  and  $b_1 = a/k$  in Eq. (9.5.3) and write it as:

$$\varphi(\rho, \phi) = \frac{Q'}{2\pi\epsilon} \ln\left(k\sqrt{\frac{\rho^2 - 2ak\rho\cos\phi + a^2k^2}{\rho^2k^2 - 2ak\rho\cos\phi + a^2}}\right) \quad (9.5.10)$$

The electric and magnetic field components are obtained from:

$$E_\rho = \eta H_\phi = -\frac{\partial\varphi}{\partial\rho}, \quad E_\phi = -\eta H_\rho = -\frac{\partial\varphi}{\rho\partial\phi} \quad (9.5.11)$$

Performing the differentiations, we find:

$$E_\rho = -\frac{Q'}{2\pi\epsilon} \left[ \frac{\rho - ak\cos\phi}{\rho^2 - 2ak\rho\cos\phi + a^2k^2} - \frac{\rho k^2 - ak\cos\phi}{\rho^2k^2 - 2ak\rho\cos\phi + a^2} \right] \quad (9.5.12)$$

$$E_\phi = -\frac{Q'}{2\pi\epsilon} \left[ \frac{ak\sin\phi}{\rho^2 - 2ak\cos\phi + a^2k^2} - \frac{ak\rho\sin\phi}{\rho^2k^2 - 2ak\rho\cos\phi + a^2} \right]$$

The resistance per unit length and corresponding attenuation constant due to conductor losses are calculated in Problem 9.3:

$$R' = \frac{R_s}{\pi a} \frac{d}{\sqrt{d^2 - 4a^2}}, \quad \alpha_c = \frac{R'}{2Z} = \frac{R_s}{2\eta a} \frac{d}{\operatorname{acosh}(d/2a)\sqrt{d^2 - 4a^2}} \quad (9.5.13)$$

## 9.6 Distributed Circuit Model of a Transmission Line

We saw that a transmission line has associated with it the parameters  $L', C'$  describing its lossless operation, and in addition, the parameters  $R', G'$  which describe the losses. It is possible then to define a series impedance  $Z'$  and a shunt admittance  $Y'$  per unit length by combining  $R'$  with  $L'$  and  $G'$  with  $C'$ :

$$\begin{aligned} Z' &= R' + j\omega L' \\ Y' &= G' + j\omega C' \end{aligned} \quad (9.6.1)$$

This leads to a so-called *distributed-parameter* circuit, which means that every infinitesimal segment  $\Delta z$  of the line can be replaced by a series impedance  $Z'\Delta z$  and a shunt admittance  $Y'\Delta z$ , as shown in Fig. 9.6.1. The voltage and current at location  $z$  will be  $V(z), I(z)$  and at location  $z + \Delta z$ ,  $V(z + \Delta z), I(z + \Delta z)$ .

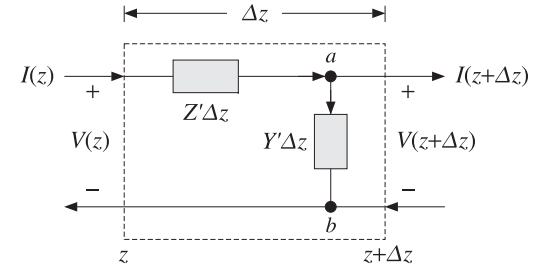


Fig. 9.6.1 Distributed parameter model of a transmission line.

The voltage across the branch  $a$ - $b$  is  $V_{ab} = V(z + \Delta z)$  and the current through it,  $I_{ab} = (Y'\Delta z)V_{ab} = Y'\Delta z V(z + \Delta z)$ . Applying Kirchhoff's voltage and current laws, we obtain:

$$\begin{aligned} V(z) &= (Z'\Delta z)I(z) + V_{ab} = Z'\Delta z I(z) + V(z + \Delta z) \\ I(z) &= I_{ab} + I(z + \Delta z) = Y'\Delta z V(z + \Delta z) + I(z + \Delta z) \end{aligned} \quad (9.6.2)$$

Using a Taylor series expansion, we may expand  $I(z + \Delta z)$  and  $V(z + \Delta z)$  to first order in  $\Delta z$ :

$$\begin{aligned} I(z + \Delta z) &= I(z) + I'(z)\Delta z \\ V(z + \Delta z) &= V(z) + V'(z)\Delta z \quad \text{and} \quad Y'\Delta z V(z + \Delta z) = Y'\Delta z V(z) \end{aligned}$$

Inserting these expressions in Eq. (9.6.2) and matching the zeroth- and first-order terms in the two sides, we obtain the equivalent differential equations:

$$\begin{aligned} V'(z) &= -Z'I(z) = -(R' + j\omega L')I(z) \\ I'(z) &= -Y'V(z) = -(G' + j\omega C')V(z) \end{aligned} \quad (9.6.3)$$

It is easily verified that the most general solution of this coupled system is expressible as a sum of a forward and a backward moving wave:

$$\begin{aligned} V(z) &= V_+ e^{-j\beta_c z} + V_- e^{j\beta_c z} \\ I(z) &= \frac{1}{Z_c} (V_+ e^{-j\beta_c z} - V_- e^{j\beta_c z}) \end{aligned} \quad (9.6.4)$$

where  $\beta_c, Z_c$  are the complex wavenumber and complex impedance:

$$\beta_c = \sqrt{-Z'Y'} = \omega \sqrt{(L' + \frac{1}{j\omega}R')(C' + \frac{1}{j\omega}G')}, \quad Z_c = \sqrt{\frac{Z'}{Y'}} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$$

The real and imaginary parts of  $\beta_c = \beta - j\alpha$  define the propagation and attenuation constants. In the case of a lossless line,  $R' = G' = 0$ , we obtain using Eq. (9.1.6):

$$\beta_c = \omega \sqrt{L'C'} = \omega \sqrt{\mu\epsilon} = \frac{\omega}{c} = \beta, \quad Z_c = \sqrt{\frac{L'}{C'}} = Z \quad (9.6.5)$$

In practice, we always assume a lossless line and then take into account the losses by assuming that  $R'$  and  $G'$  are small quantities, which can be evaluated by the appropriate expressions that can be derived for each type of line, as we did for the parallel-plate, coaxial, and two-wire lines. The lossless solution (9.6.4) takes the form:

$$\begin{aligned} V(z) &= V_+ e^{-j\beta z} + V_- e^{j\beta z} = V_+(z) + V_-(z) \\ I(z) &= \frac{1}{Z}(V_+ e^{-j\beta z} - V_- e^{j\beta z}) = \frac{1}{Z}(V_+(z) - V_-(z)) \end{aligned} \quad (9.6.6)$$

This solution is identical to that of uniform plane waves of Chap. 4, provided we make the identifications:

$$\begin{array}{|l} V(z) \longleftrightarrow E(z) \\ I(z) \longleftrightarrow H(z) \\ Z \longleftrightarrow \eta \end{array} \quad \text{and} \quad \begin{array}{|l} V_+(z) \longleftrightarrow E_+(z) \\ V_-(z) \longleftrightarrow E_-(z) \end{array}$$

### 9.7 Wave Impedance and Reflection Response

All the concepts of Chap. 4 translate verbatim to the transmission line case. For example, we may define the *wave impedance* and *reflection response* at location  $z$ :

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{V_+(z) + V_-(z)}{V_+(z) - V_-(z)}, \quad \Gamma(z) = \frac{V_-(z)}{V_+(z)} \quad (9.7.1)$$

To avoid ambiguity in notation, we will denote the characteristic impedance of the line by  $Z_0$ . It follows from Eq. (9.7.1) that  $Z(z)$  and  $\Gamma(z)$  are related by:

$$\boxed{Z(z) = Z_0 \frac{1 + \Gamma(z)}{1 - \Gamma(z)}}, \quad \boxed{\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}} \quad (9.7.2)$$

For a forward-moving wave, the conditions  $\Gamma(z) = 0$  and  $Z(z) = Z_0$  are equivalent. The propagation equations of  $Z(z)$  and  $\Gamma(z)$  between two points  $z_1, z_2$  along the line separated by distance  $l = z_2 - z_1$  are given by:

$$\boxed{Z_1 = Z_0 \frac{Z_2 + jZ_0 \tan \beta l}{Z_0 + jZ_1 \tan \beta l}} \Leftrightarrow \boxed{\Gamma_1 = \Gamma_2 e^{-2j\beta l}} \quad (9.7.3)$$

where we have the relationships between  $Z_1, Z_2$  and  $\Gamma_1, \Gamma_2$ :

$$Z_1 = Z_0 \frac{1 + \Gamma_1}{1 - \Gamma_1}, \quad Z_2 = Z_0 \frac{1 + \Gamma_2}{1 - \Gamma_2} \quad (9.7.4)$$

We may also express  $Z_1$  in terms of  $\Gamma_2$ :

$$\boxed{Z_1 = Z_0 \frac{1 + \Gamma_1}{1 - \Gamma_1} = Z_0 \frac{1 + \Gamma_2 e^{-2j\beta l}}{1 - \Gamma_2 e^{-2j\beta l}}} \quad (9.7.5)$$

The relationship between the voltage and current waves at points  $z_1$  and  $z_2$  is obtained by the propagation matrix:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \cos \beta l & jZ_0 \sin \beta l \\ jZ_0^{-1} \sin \beta l & \cos \beta l \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad (\text{propagation matrix}) \quad (9.7.6)$$

Similarly, we may relate the forward/backward voltages at the points  $z_1$  and  $z_2$ :

$$\begin{bmatrix} V_{1+} \\ V_{1-} \end{bmatrix} = \begin{bmatrix} e^{j\beta l} & 0 \\ 0 & e^{-j\beta l} \end{bmatrix} \begin{bmatrix} V_{2+} \\ V_{2-} \end{bmatrix} \quad (\text{propagation matrix}) \quad (9.7.7)$$

It follows from Eq. (9.6.6) that  $V_{1\pm}, V_{2\pm}$  are related to  $V_1, I_1$  and  $V_2, I_2$  by:

$$V_{1\pm} = \frac{1}{2}(V_1 \pm Z_0 I_1), \quad V_{2\pm} = \frac{1}{2}(V_2 \pm Z_0 I_2) \quad (9.7.8)$$

Fig. 9.7.1 depicts these various quantities. We note that the behavior of the line remains unchanged if the line is cut at the point  $z_2$  and the entire right portion of the line is replaced by an impedance equal to  $Z_2$ , as shown in the figure.

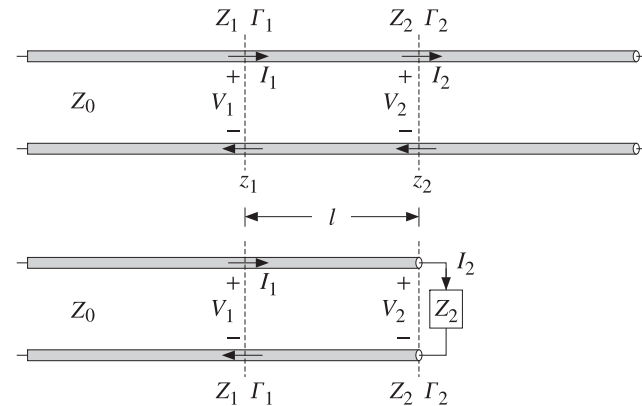


Fig. 9.7.1 Length segment on infinite line and equivalent terminated line.

This is so because in both cases, all the points  $z_1$  to the left of  $z_2$  see the same voltage-current relationship at  $z_2$ , that is,  $V_2 = Z_2 I_2$ .

Sometimes, as in the case of designing stub tuners for matching a line to a load, it is more convenient to work with the *wave admittances*. Defining  $Y_0 = 1/Z_0$ ,  $Y_1 = 1/Z_1$ , and  $Y_2 = 1/Z_2$ , it is easily verified that the admittances satisfy exactly the same propagation relationship as the impedances:

$$Y_1 = Y_0 \frac{Y_2 + jY_0 \tan \beta l}{Y_0 + jY_2 \tan \beta l} \quad (9.7.9)$$

As in the case of dielectric slabs, the half- and quarter-wavelength separations are of special interest. For a half-wave distance, we have  $\beta l = 2\pi/2 = \pi$ , which translates to  $l = \lambda/2$ , where  $\lambda = 2\pi/\beta$  is the wavelength along the line. For a quarter-wave, we have  $\beta l = 2\pi/4 = \pi/2$  or  $l = \lambda/4$ . Setting  $\beta l = \pi$  or  $\pi/2$  in Eq. (9.7.3), we obtain:

$$\begin{aligned} l = \frac{\lambda}{2} &\Rightarrow Z_1 = Z_2, \quad \Gamma_1 = \Gamma_2 \\ l = \frac{\lambda}{4} &\Rightarrow Z_1 = \frac{Z_0^2}{Z_2}, \quad \Gamma_1 = -\Gamma_2 \end{aligned} \quad (9.7.10)$$

The MATLAB functions `z2g.m` and `g2z.m` compute  $\Gamma$  from  $Z$  and conversely, by implementing Eq. (9.7.2). The functions `gprop.m`, `zprop.m` and `vprop.m` implement the propagation equations (9.7.3) and (9.7.6). The usage of these functions is:

```
G = z2g(Z,Z0);           % Z to Γ
Z = g2z(G,Z0);           % Γ to Z
G1 = gprop(G2,b1);       % propagates Γb2 to Γb1
Z1 = zprop(Z2,Z0,b1);    % propagates Zb2 to Zb1
[V1,I1] = vprop(V2,I2,Z0,b1); % propagates Vb2,Ib2 to Vb1,Ib1
```

The parameter `b1` is  $\beta l$ . The propagation equations and these MATLAB functions also work for lossy lines. In this case,  $\beta$  must be replaced by the complex wavenumber  $\beta_c = \beta - j\alpha$ . The propagation phase factors become now:

$$e^{\pm j\beta l} \rightarrow e^{\pm j\beta_c l} = e^{\pm \alpha l} e^{\pm j\beta l} \quad (9.7.11)$$

## 9.8 Two-Port Equivalent Circuit

Any length- $l$  segment of a transmission line may be represented as a two-port equivalent circuit. Rearranging the terms in Eq. (9.7.6), we may write it in *impedance-matrix* form:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ -I_2 \end{bmatrix} \quad (\text{impedance matrix}) \quad (9.8.1)$$

where the impedance elements are:

$$\begin{aligned} Z_{11} &= Z_{22} = -jZ_0 \cot \beta l \\ Z_{12} &= Z_{21} = -jZ_0 \frac{1}{\sin \beta l} \end{aligned} \quad (9.8.2)$$

The negative sign,  $-I_2$ , conforms to the usual convention of having the currents coming *into* the two-port from either side. This impedance matrix can also be realized in a *T-section* configuration as shown in Fig. 9.8.1.

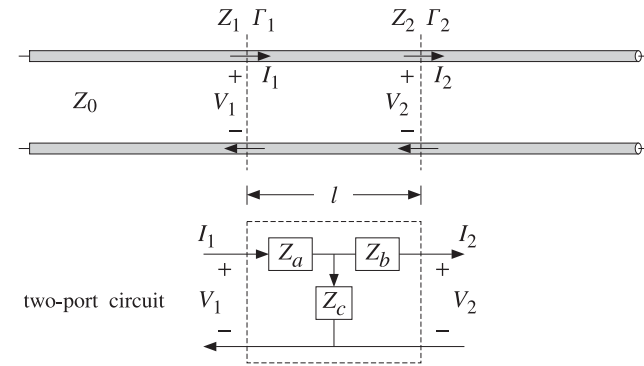


Fig. 9.8.1 Length- $l$  segment of a transmission line and its equivalent *T-section*.

Using Eq. (9.8.1) and some trigonometry, the impedances  $Z_a, Z_b, Z_c$  of the *T-section* are found to be:

$$\begin{aligned} Z_a &= Z_{11} - Z_{12} = jZ_0 \tan(\beta l/2) \\ Z_b &= Z_{22} - Z_{12} = jZ_0 \tan(\beta l/2) \\ Z_c &= Z_{12} = -jZ_0 \frac{1}{\sin \beta l} \end{aligned} \quad (9.8.3)$$

The MATLAB function `tsection.m` implements Eq. (9.8.3). Its usage is:

```
[Za,Zc] = tsection(Z0,b1);
```

## 9.9 Terminated Transmission Lines

We can use the results of the previous section to analyze the behavior of a transmission line connected between a generator and a load. For example in a transmitting antenna system, the transmitter is the generator and the antenna, the load. In a receiving system, the antenna is the generator and the receiver, the load.

Fig. 9.9.1 shows a generator of voltage  $V_G$  and internal impedance  $Z_G$  connected to the load impedance  $Z_L$  through a length  $d$  of a transmission line of characteristic impedance  $Z_0$ . We wish to determine the voltage and current at the load in terms of the generator voltage.

We assume that the line is lossless and hence  $Z_0$  is real. The generator impedance is also assumed to be real but it does not have to be. The load impedance will have in general both a resistive and a reactive part,  $Z_L = R_L + jX_L$ .

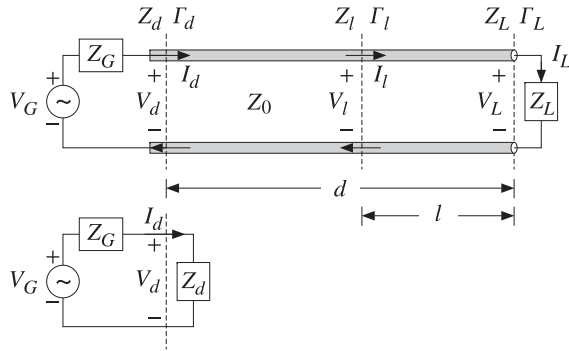


Fig. 9.9.1 Terminated line and equivalent circuit.

At the load location, the voltage, current, and impedance are  $V_L, I_L, Z_L$  and play the same role as the quantities  $V_2, I_2, Z_2$  of the previous section. They are related by  $V_L = Z_L I_L$ . The reflection coefficient at the load will be:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \Leftrightarrow Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \quad (9.9.1)$$

The quantities  $Z_L, \Gamma_L$  can be propagated now by a distance  $d$  to the generator at the input to the line. The corresponding voltage, current, and impedance  $V_d, I_d, Z_d$  play the role of  $V_1, I_1, Z_1$  of the previous section, and are related by  $V_d = Z_d I_d$ . We have the propagation relationships:

$$Z_d = Z_0 \frac{Z_L + jZ_0 \tan \beta d}{Z_0 + jZ_L \tan \beta d} \Leftrightarrow \Gamma_d = \Gamma_L e^{-2j\beta d} \quad (9.9.2)$$

where

$$\Gamma_d = \frac{Z_d - Z_0}{Z_d + Z_0} \Leftrightarrow Z_d = Z_0 \frac{1 + \Gamma_d}{1 - \Gamma_d} = Z_0 \frac{1 + \Gamma_L e^{-2j\beta d}}{1 - \Gamma_L e^{-2j\beta d}} \quad (9.9.3)$$

At the line input, the entire length- $d$  line segment and load can be replaced by the impedance  $Z_d$ , as shown in Fig. 9.9.1. We have now a simple voltage divider circuit. Thus,

$$V_d = V_G - I_d Z_G = \frac{V_G Z_d}{Z_G + Z_d}, \quad I_d = \frac{V_G}{Z_G + Z_d} \quad (9.9.4)$$

Once we have  $V_d, I_d$  in terms of  $V_G$ , we can invert the propagation matrix (9.7.6) to obtain the voltage and current at the load:

$$\begin{bmatrix} V_L \\ I_L \end{bmatrix} = \begin{bmatrix} \cos \beta d & -jZ_0 \sin \beta d \\ -jZ_0^{-1} \sin \beta d & \cos \beta d \end{bmatrix} \begin{bmatrix} V_d \\ I_d \end{bmatrix} \quad (9.9.5)$$

It is more convenient to express  $V_d, I_d$  in terms of the reflection coefficients  $\Gamma_d$  and  $\Gamma_G$ , the latter being defined by:

$$\Gamma_G = \frac{Z_G - Z_0}{Z_G + Z_0} \Leftrightarrow Z_G = Z_0 \frac{1 + \Gamma_G}{1 - \Gamma_G} \quad (9.9.6)$$

It is easy to verify using Eqs. (9.9.3) and (9.9.6) that:

$$Z_G + Z_d = 2Z_0 \frac{1 - \Gamma_G \Gamma_d}{(1 - \Gamma_G)(1 - \Gamma_d)}, \quad Z_G + Z_0 = 2Z_0 \frac{1}{1 - \Gamma_G}$$

From these, it follows that:

$$V_d = \frac{V_G Z_0}{Z_G + Z_0} \frac{1 + \Gamma_d}{1 - \Gamma_G \Gamma_d}, \quad I_d = \frac{V_G}{Z_G + Z_0} \frac{1 - \Gamma_d}{1 - \Gamma_G \Gamma_d} \quad (9.9.7)$$

where  $\Gamma_d$  may be replaced by  $\Gamma_d = \Gamma_L e^{-2j\beta d}$ . If the line and load are matched so that  $Z_L = Z_0$ , then  $\Gamma_L = 0$  and  $\Gamma_d = 0$  and  $Z_d = Z_0$  for any distance  $d$ . Eq. (9.9.7) then reduces to:

$$V_d = \frac{V_G Z_0}{Z_G + Z_0}, \quad I_d = \frac{V_G}{Z_G + Z_0} \quad (\text{matched load}) \quad (9.9.8)$$

In this case, there is only a forward-moving wave along the line. The voltage and current at the load will correspond to the propagation of these quantities to location  $l = 0$ , which introduces a propagation phase factor  $e^{-j\beta d}$ :

$$V_0 = \frac{V_G Z_0}{Z_G + Z_0} e^{-j\beta d}, \quad I_0 = \frac{V_G}{Z_G + Z_0} e^{-j\beta d} \quad (\text{matched load}) \quad (9.9.9)$$

where  $V_0, I_0$  denote  $V_L, I_L$  when  $Z_L = Z_0$ . It is convenient also to express  $V_L$  directly in terms of  $V_d$  and the reflection coefficients  $\Gamma_d$  and  $\Gamma_L$ . We note that:

$$V_L = V_{L+}(1 + \Gamma_L), \quad V_{L+} = V_{d+} e^{-j\beta d}, \quad V_{d+} = \frac{V_d}{1 + \Gamma_d}$$

It follows that the voltage  $V_L$  and current  $I_L = V_L / Z_L$  are:

$$V_L = V_d e^{-j\beta d} \frac{1 + \Gamma_L}{1 + \Gamma_d}, \quad I_L = I_d e^{-j\beta d} \frac{1 - \Gamma_L}{1 - \Gamma_d} \quad (9.9.10)$$

Expressing  $V_L$  and also  $I_L = V_L / Z_L$  directly in terms of  $V_G$ , we have:

$$V_L = \frac{V_G Z_0}{Z_G + Z_0} \frac{1 + \Gamma_L}{1 - \Gamma_G \Gamma_d} e^{-j\beta d}, \quad I_L = \frac{V_G}{Z_G + Z_0} \frac{1 - \Gamma_L}{1 - \Gamma_G \Gamma_d} e^{-j\beta d} \quad (9.9.11)$$

It should be emphasized that  $d$  refers to the fixed distance between the generator and the load. For any other distance, say  $l$ , from the load (or, distance  $z = d - l$  from the generator,) the voltage and current can be expressed in terms of the load voltage and current as follows:

$$V_l = V_L e^{j\beta l} \frac{1 + \Gamma_l}{1 + \Gamma_L}, \quad I_l = I_L e^{j\beta l} \frac{1 - \Gamma_l}{1 - \Gamma_L}, \quad \Gamma_l = \Gamma_L e^{-2j\beta l} \quad (9.9.12)$$

### 9.10 Power Transfer from Generator to Load

The *total power* delivered by the generator is dissipated partly in its internal resistance and partly in the load. The power delivered to the load is equal (for a lossless line) to the net power traveling to the right at any point along the line. Thus, we have:

$$P_{\text{tot}} = P_d + P_G = P_L + P_G \quad (9.10.1)$$

This follows from  $V_G = V_d + I_d Z_G$ , which implies

$$V_G I_d^* = V_d I_d^* + Z_G |I_d|^2 \quad (9.10.2)$$

Eq. (9.10.1) is a consequence of (9.10.2) and the definitions:

$$\begin{aligned} P_{\text{tot}} &= \frac{1}{2} \text{Re}(V_G^* I_d) = \frac{1}{2} \text{Re}[(V_d + Z_G I_d)^* I_d] \\ P_G &= \frac{1}{2} \text{Re}(Z_G I_d I_d^*) = \frac{1}{2} \text{Re}(Z_G) |I_d|^2 \\ P_d &= \frac{1}{2} \text{Re}(V_d^* I_d) = \frac{1}{2} \text{Re}(V_L^* I_L) = P_L \end{aligned} \quad (9.10.3)$$

The last equality follows from Eq. (9.9.5) or from  $V_{d\pm} = V_{L\pm} e^{\pm j\beta d}$ :

$$\frac{1}{2} \text{Re}(V_d^* I_d) = \frac{1}{2Z_0} (|V_{d+}|^2 - |V_{d-}|^2) = \frac{1}{2Z_0} (|V_{L+}|^2 - |V_{L-}|^2) = \frac{1}{2} \text{Re}(V_L^* I_L)$$

In the special case when the generator and the load are matched to the line, so that  $Z_G = Z_L = Z_0$ , then we find the standard result that half of the generated power is delivered to the load and half is lost in the internal impedance. Using Eq. (9.9.8) with  $Z_G = Z_0$ , we obtain  $V_d = I_d Z_G = V_G/2$ , which gives:

$$P_{\text{tot}} = \frac{|V_G|^2}{4Z_0}, \quad P_G = \frac{|V_G|^2}{8Z_0} = \frac{1}{2} P_{\text{tot}}, \quad P_d = P_L = \frac{|V_G|^2}{8Z_0} = \frac{1}{2} P_{\text{tot}} \quad (9.10.4)$$

**Example 9.10.1:** A load  $Z_L = 50 + j10 \Omega$  is connected to a generator  $V_G = 10 \angle 0^\circ$  volts with a 100-ft (30.48 m) cable of a 50-ohm transmission line. The generator's internal impedance is 20 ohm, the operating frequency is 10 MHz, and the velocity factor of the line,  $2/3$ .

Determine the voltage across the load, the total power delivered by the generator, the power dissipated in the generator's internal impedance and in the load.

**Solution:** The propagation speed is  $c = 2c_0/3 = 2 \times 10^8$  m/sec. The line wavelength  $\lambda = c/f = 20$  m and the propagation wavenumber  $\beta = 2\pi/\lambda = 0.3142$  rads/m. The electrical length is  $d/\lambda = 30.48/20 = 1.524$  and the phase length  $\beta d = 9.5756$  radians.

Next, we calculate the reflection coefficients:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = 0.0995 \angle 84.29^\circ, \quad \Gamma_G = \frac{Z_G - Z_0}{Z_G + Z_0} = -0.4286$$

and  $\Gamma_d = \Gamma_L e^{-2j\beta d} = 0.0995 \angle 67.01^\circ$ . It follows that:

$$Z_d = Z_0 \frac{1 + \Gamma_d}{1 - \Gamma_d} = 53.11 + j9.83, \quad V_d = \frac{V_G Z_d}{Z_G + Z_d} = 7.31 + j0.36 = 7.32 \angle 2.83^\circ$$

The voltage across the load will be:

$$V_L = V_d e^{-j\beta d} \frac{1 + \Gamma_L}{1 + \Gamma_d} = -7.09 + j0.65 = 7.12 \angle 174.75^\circ \text{ V}$$

The current through the generator is:

$$I_d = \frac{V_d}{Z_d} = 0.13 + j0.02 = 0.14 \angle -7.66^\circ \text{ A}$$

It follows that the generated and dissipated powers will be:

$$\begin{aligned} P_{\text{tot}} &= \frac{1}{2} \text{Re}(V_G^* I_d) = 0.6718 \text{ W} \\ P_G &= \frac{1}{2} \text{Re}(Z_G) |I_d|^2 = 0.1388 \text{ W} \\ P_L = P_d &= \frac{1}{2} \text{Re}(V_d^* I_d) = 0.4880 \text{ W} \end{aligned}$$

We note that  $P_{\text{tot}} = P_G + P_L$ . □

If the line is lossy, with a complex wavenumber  $\beta_c = \beta - j\alpha$ , the power  $P_L$  at the output of the line is less than the power  $P_d$  at the input of the line. Writing  $V_{d\pm} = V_{L\pm} e^{\pm \alpha d} e^{\pm j\beta d}$ , we find:

$$\begin{aligned} P_d &= \frac{1}{2Z_0} (|V_{d+}|^2 - |V_{d-}|^2) = \frac{1}{2Z_0} (|V_{L+}|^2 e^{2\alpha d} - |V_{L-}|^2 e^{-2\alpha d}) \\ P_L &= \frac{1}{2Z_0} (|V_{L+}|^2 - |V_{L-}|^2) \end{aligned}$$

We note that  $P_d > P_L$  for all  $\Gamma_L$ . In terms of the incident forward power at the load,  $P_{\text{inc}} = |V_{L+}|^2 / 2Z_0$ , we have:

$$\begin{aligned} P_d &= P_{\text{inc}} (e^{2\alpha d} - |\Gamma_L|^2 e^{-2\alpha d}) = P_{\text{inc}} e^{2\alpha d} (1 - |\Gamma_d|^2) \\ P_L &= P_{\text{inc}} (1 - |\Gamma_L|^2) \end{aligned} \quad (9.10.5)$$

where  $|\Gamma_d| = |\Gamma_L| e^{-\alpha d}$ . The total attenuation or loss of the line is  $P_d/P_L$  (the inverse  $P_L/P_d$  is the total gain, which is less than one.) In decibels, the loss is:

$$L = 10 \log_{10} \left( \frac{P_d}{P_L} \right) = 10 \log_{10} \left( \frac{e^{2\alpha d} - |\Gamma_L|^2 e^{-2\alpha d}}{1 - |\Gamma_L|^2} \right) \quad (\text{total loss}) \quad (9.10.6)$$

If the load is matched to the line,  $Z_L = Z_0$ , so that  $\Gamma_L = 0$ , the loss is referred to as the *matched-line* loss and is due only to the transmission losses along the line:

$$L_M = 10 \log_{10}(e^{2\alpha d}) = 8.686\alpha d \quad (\text{matched-line loss}) \quad (9.10.7)$$

Denoting the matched-line loss in absolute units by  $a = 10^{L_M/10} = e^{2\alpha d}$ , we may write Eq. (9.10.6) in the equivalent form:

$$L = 10 \log_{10} \left( \frac{a^2 - |\Gamma_L|^2}{a(1 - |\Gamma_L|^2)} \right) \quad (\text{total loss}) \quad (9.10.8)$$

The additional loss due to the mismatched load is the difference:

$$L - L_M = 10 \log_{10} \left( \frac{1 - |\Gamma_L|^2 e^{-4\alpha d}}{1 - |\Gamma_L|^2} \right) = 10 \log_{10} \left( \frac{1 - |\Gamma_d|^2}{1 - |\Gamma_L|^2} \right) \quad (9.10.9)$$

**Example 9.10.2:** A 150 ft long RG-58 coax is connected to a load  $Z_L = 25 + 50j$  ohm. At the operating frequency of 10 MHz, the cable is rated to have 1.2 dB/100 ft of matched-line loss. Determine the total loss of the line and the excess loss due to the mismatched load.

**Solution:** The matched-line loss of the 150 ft cable is  $L_M = 150 \times 1.2/100 = 1.8$  dB or in absolute units,  $a = 10^{1.8/10} = 1.51$ . The reflection coefficient has magnitude computed with the help of the MATLAB function `z2g`:

$$|\Gamma_L| = \text{abs}(z2g(25 + 50j, 50)) = 0.62$$

It follows that the total loss will be:

$$L = 10 \log_{10} \left( \frac{a^2 - |\Gamma_L|^2}{a(1 - |\Gamma_L|^2)} \right) = 10 \log_{10} \left( \frac{1.51^2 - 0.62^2}{1.51(1 - 0.62^2)} \right) = 3.1 \text{ dB}$$

The excess loss due to the mismatched load is  $3.1 - 1.8 = 1.3$  dB. At the line input, we have  $|\Gamma_d| = |\Gamma_L|e^{-2\alpha d} = |\Gamma_L|/a = 0.62/1.51 = 0.41$ . Therefore, from the point of view of the input the line appears to be more matched.  $\square$

### 9.11 Open- and Short-Circuited Transmission Lines

Open- and short-circuited transmission lines are widely used to construct resonant circuits as well as matching stubs. They correspond to the special cases for the load impedance:  $Z_L = \infty$  for an open-circuited line and  $Z_L = 0$  for a short-circuited one. Fig. 9.11.1 shows these two cases.

Knowing the open-circuit voltage and the short-circuit current at the end terminals  $a, b$ , allows us also to replace the entire left segment of the line, including the generator, with a Thévenin-equivalent circuit. Connected to a load impedance  $Z_L$ , the equivalent circuit will produce the same load voltage and current  $V_L, I_L$  as the original line and generator.

Setting  $Z_L = \infty$  and  $Z_L = 0$  in Eq. (9.9.2), we obtain the following expressions for the wave impedance  $Z_l$  at distance  $l$  from the open- or short-circuited termination:

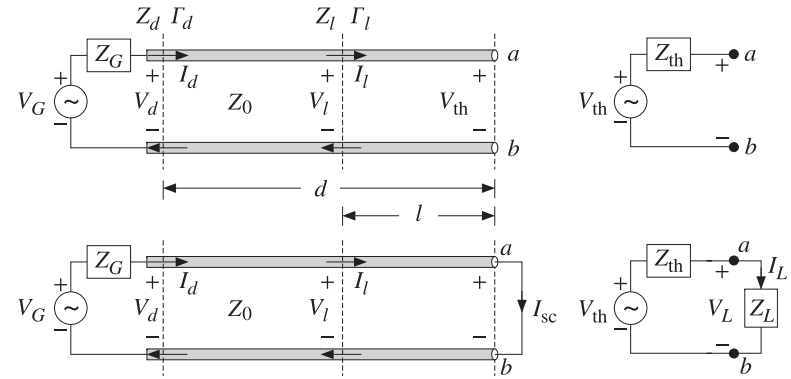


Fig. 9.11.1 Open- and short-circuited line and Thévenin-equivalent circuit.

$$\begin{aligned} Z_l &= -jZ_0 \cot \beta l && (\text{open-circuited}) \\ Z_l &= jZ_0 \tan \beta l && (\text{short-circuited}) \end{aligned} \quad (9.11.1)$$

The corresponding admittances  $Y_l = 1/Z_l$  will be:

$$\begin{aligned} Y_l &= jY_0 \tan \beta l && (\text{open-circuited}) \\ Y_l &= -jY_0 \cot \beta l && (\text{short-circuited}) \end{aligned} \quad (9.11.2)$$

To determine the Thévenin-equivalent circuit that replaces everything to the left of the terminals  $a, b$ , we must find the open-circuit voltage  $V_{th}$ , the short-circuit current  $I_{sc}$ , and the Thévenin impedance  $Z_{th}$ .

The impedance  $Z_{th}$  can be determined either by  $Z_{th} = V_{th}/I_{sc}$ , or by disconnecting the generator and finding the equivalent impedance looking to the left of the terminals  $a, b$ . It is obtained by propagating the generator impedance  $Z_G$  by a distance  $d$ :

$$Z_{th} = Z_0 \frac{Z_G + jZ_0 \tan \beta d}{Z_0 + jZ_G \tan \beta d} = Z_0 \frac{1 + \Gamma_{th}}{1 - \Gamma_{th}}, \quad \Gamma_{th} = \Gamma_G e^{-2j\beta d} \quad (9.11.3)$$

The open-circuit voltage can be determined from Eq. (9.9.11) by setting  $Z_L = \infty$ , which implies that  $\Gamma_L = 1$ ,  $\Gamma_d = e^{-2j\beta d}$ , and  $\Gamma_G \Gamma_d = \Gamma_G e^{-2j\beta d} = \Gamma_{th}$ . The short-circuit current is also obtained from (9.9.11) by setting  $Z_L = 0$ , which gives  $\Gamma_L = -1$ ,  $\Gamma_d = -e^{-2j\beta d}$ , and  $\Gamma_G \Gamma_d = -\Gamma_G e^{-2j\beta d} = -\Gamma_{th}$ . Then, we find:

$$V_{th} = \frac{V_G Z_0}{Z_G + Z_0} \frac{2e^{-j\beta d}}{1 - \Gamma_{th}}, \quad I_{sc} = \frac{V_G}{Z_G + Z_0} \frac{2e^{-j\beta d}}{1 + \Gamma_{th}} \quad (9.11.4)$$

It follows that  $V_{th}/I_{sc} = Z_{th}$ , as given by Eq. (9.11.3). A more convenient way of writing Eq. (9.11.4) is by noting the relationships:

$$1 - \Gamma_{th} = \frac{2Z_0}{Z_{th} + Z_0}, \quad 1 + \Gamma_{th} = \frac{2Z_{th}}{Z_{th} + Z_0}$$

Then, Eq. (9.11.4) becomes:

$$V_{th} = V_0 \frac{Z_{th} + Z_0}{Z_0}, \quad I_{sc} = I_0 \frac{Z_{th} + Z_0}{Z_{th}} \tag{9.11.5}$$

where  $V_0, I_0$  are the load voltage and currents in the matched case, given by Eq. (9.9.9). The intuitive meaning of these expressions can be understood by writing them as:

$$V_0 = V_{th} \frac{Z_0}{Z_{th} + Z_0}, \quad I_0 = I_{sc} \frac{Z_{th}}{Z_{th} + Z_0} \tag{9.11.6}$$

These are recognized to be the ordinary voltage and current dividers obtained by connecting the Thévenin and Norton equivalent circuits to the matched load impedance  $Z_0$ , as shown in Fig. 9.11.2.

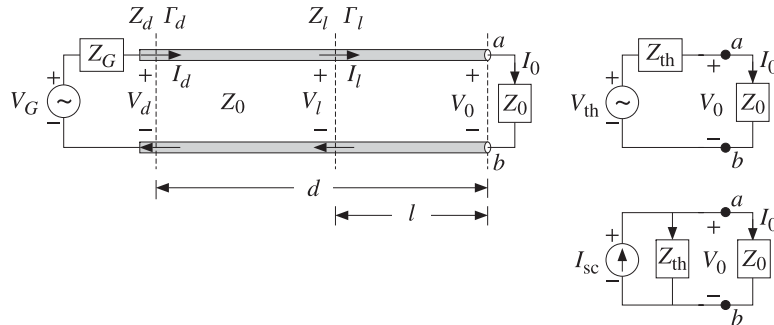


Fig. 9.11.2 Thévenin and Norton equivalent circuits connected to a matched load.

The quantities  $V_0, I_0$  are the same as those obtained by connecting the actual line to the matched load, as was done in Eq. (9.9.9).

An alternative way of determining the quantities  $V_{th}$  and  $Z_{th}$  is by replacing the length- $d$  transmission line segment by its  $T$ -section equivalent circuit, as shown in Fig. 9.11.3.

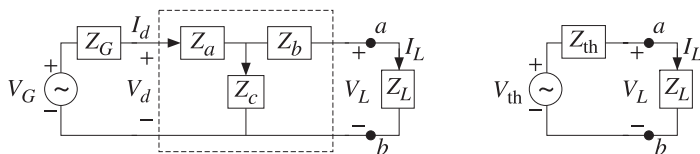


Fig. 9.11.3  $T$ -section and Thévenin equivalent circuits.

The Thévenin equivalent circuit to the left of the terminals  $a, b$  is easily determined by shorting the generator and finding the Thévenin impedance and then finding the open-circuit voltage. We have:

$$Z_{th} = Z_b + \frac{Z_c(Z_a + Z_G)}{Z_c + Z_a + Z_G}, \quad V_{th} = \frac{V_G Z_c}{Z_c + Z_a + Z_G} \tag{9.11.7}$$

where  $Z_a, Z_b, Z_c$  for a length- $d$  segment are given by Eq. (9.8.3):

$$Z_a = Z_b = jZ_0 \tan\left(\frac{\beta d}{2}\right), \quad Z_c = -jZ_0 \frac{1}{\sin \beta d}$$

It is straightforward to verify that the expressions in Eq. (9.11.7) are equivalent to those in Eq. (9.11.3) and (9.11.4).

**Example 9.11.1:** For the generator, line, and load of Example 9.10.1, determine the Thévenin equivalent circuit. Using this circuit determine the load voltage.

**Solution:** We work with the  $T$ -section approach. The following MATLAB call gives  $Z_a$  and  $Z_c$ , with  $Z_0 = 50$  and  $\beta d = 9.5756$ :

$$[Z_a, Z_c] = \text{tsection}(50, 9.5756) = [-661.89j, 332.83j]$$

Then, Eq. (9.11.7) gives with  $Z_b = Z_a$ :

$$Z_{th} = Z_b + \frac{Z_c(Z_a + Z_G)}{Z_c + Z_a + Z_G} = 20.39 + j6.36 \Omega$$

$$V_{th} = \frac{V_G Z_c}{Z_c + Z_a + Z_G} = -10.08 + j0.61 = 10.10 \angle 176.52^\circ \text{ V}$$

Alternatively,  $Z_{th}$  can be computed by propagating  $Z_G = 20$  by a distance  $d$ :

$$Z_{th} = \text{zprop}(20, 50, 9.5756) = 20.39 + j6.36 \Omega$$

The load voltage is found from the Thévenin circuit:

$$V_L = \frac{V_{th} Z_L}{Z_L + Z_{th}} = -7.09 + j0.65 = 7.12 \angle 174.75^\circ \text{ V}$$

which agrees with that found in Example 9.10.1. □

### 9.12 Standing Wave Ratio

The line voltage at a distance  $l$  from the load is given by Eq. (9.9.12), which can be written as follows in terms of the forward wave  $V_{L+} = V_L / (1 + \Gamma_L)$ :

$$V_l = V_{L+} e^{j\beta l} (1 + \Gamma_l) \tag{9.12.1}$$

The magnitude of  $V_l$  will be:



$$|V_l| = |V_{L+}| |1 + \Gamma_l| = |V_{L+}| |1 + \Gamma_L e^{-2j\beta l}| \quad (9.12.2)$$

It follows that  $|V_l|$  will vary sinusoidally as a function of  $l$ . Its limits of variation are determined by noting that the quantity  $|1 + \Gamma_l|$  varies between:

$$1 - |\Gamma_L| = 1 - |\Gamma_l| \leq |1 + \Gamma_l| \leq 1 + |\Gamma_l| = 1 + |\Gamma_L|$$

where we used  $|\Gamma_l| = |\Gamma_L|$ . Thus,  $|V_l|$  will vary over the limits:

$$V_{\min} \leq |V_l| \leq V_{\max} \quad (9.12.3)$$

where

$$V_{\min} = |V_{L+}| - |V_{L-}| = |V_{L+}| (1 - |\Gamma_L|) \quad (9.12.4)$$

$$V_{\max} = |V_{L+}| + |V_{L-}| = |V_{L+}| (1 + |\Gamma_L|)$$

We note that the reflection coefficient at a load  $Z_L = R_L + jX_L$  has always magnitude less than unity,  $|\Gamma_L| \leq 1$ . Indeed, this follows from the positivity of  $R_L$  and the following property:

$$Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \Rightarrow R_L = \text{Re}(Z_L) = Z_0 \frac{1 - |\Gamma_L|^2}{|1 + \Gamma_L|^2} \quad (9.12.5)$$

The voltage *standing wave ratio* (SWR) of a terminated transmission line is a measure of the degree of matching of the line to the load and is defined as the ratio of the maximum to minimum voltage along the line:

$$S = \frac{V_{\max}}{V_{\min}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \Leftrightarrow |\Gamma_L| = \frac{S - 1}{S + 1} \quad (9.12.6)$$

Because  $|\Gamma_L| \leq 1$ , the SWR will always be  $S \geq 1$ . A matched load,  $\Gamma_L = 0$ , has  $S = 1$ . The more unmatched the load is, the larger the SWR. Indeed,  $S \rightarrow \infty$  as  $|\Gamma_L| \rightarrow 1$ . A matched line has  $V_{\min} = |V_l| = V_{\max}$  at all points  $l$ , and is sometimes referred to as a *flat line*. The MATLAB function `swr.m` calculates the SWR from Eq. (9.12.6):

$$S = \text{swr}(\text{Gamma}); \quad \% \text{ calculates SWR from reflection coefficient } \Gamma$$

The SWR can be used to quantify the amount of power delivered to the load. The percentage of reflected power from the load is  $|\Gamma_L|^2$ . Therefore, the percentage of the power delivered to the load relative to the incident power will be:

$$\frac{P_L}{P_{\text{inc}}} = 1 - |\Gamma_L|^2 = \frac{4S}{(S + 1)^2} \quad (9.12.7)$$

The larger the SWR, the smaller the percentage of delivered power. For example, if  $S = 9$ , the reflection coefficient will have magnitude  $|\Gamma_L| = 0.8$ , resulting in  $1 - |\Gamma_L|^2 = 0.36$ , that is, only 36 percent of the incident power gets transferred to the load.

**Example 9.12.1:** If the reflected wave at the load of a transmission line is 6 dB below the incident wave, what is the SWR at the load? What percentage of the incident power gets transferred to the load?

**Solution:** The relative power levels of the reflected and incident waves will be:

$$|\Gamma_L|^2 = \frac{|V_-|^2}{|V_+|^2} = 10^{-6/10} = \frac{1}{4} \Rightarrow |\Gamma_L| = \frac{1}{2} \Rightarrow S = \frac{1 + 0.5}{1 - 0.5} = 3$$

The fraction of power transferred to the load is  $1 - |\Gamma_L|^2 = 0.75$ , or 75 percent.  $\square$

If both the line and load impedances are real-valued, then the standing wave ratio is  $S = Z_L/Z_0$  if  $Z_L \geq Z_0$ , and  $S = Z_0/Z_L$ , if  $Z_L \leq Z_0$ . This follows from the identity:

$$S = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \frac{|Z_L + Z_0| + |Z_L - Z_0|}{|Z_L + Z_0| - |Z_L - Z_0|} = \frac{\max(Z_L, Z_0)}{\min(Z_L, Z_0)} \quad (9.12.8)$$

or, explicitly:

$$S = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \begin{cases} \frac{Z_L}{Z_0}, & \text{if } Z_L \geq Z_0 \\ \frac{Z_0}{Z_L}, & \text{if } Z_L \leq Z_0 \end{cases} \quad (9.12.9)$$

### 9.13 Determining an Unknown Load Impedance

Often a transmission line is connected to an unknown impedance, and we wish to determine that impedance by making appropriate measurements of the voltage along the line.

The SWR can be readily determined by measuring  $|V_l|$  and finding its maximum and minimum values  $V_{\max}$  and  $V_{\min}$ . From the SWR, we then determine the magnitude of the reflection coefficient  $|\Gamma_L|$ .

The phase of  $\Gamma_L$  can be determined by finding the locations along the line at which a voltage maximum or a voltage minimum is measured. If  $\theta_L$  is the required phase, so that  $\Gamma_L = |\Gamma_L| e^{j\theta_L}$ , then we have:

$$|V_l| = |V_{L+}| |1 + \Gamma_l| = |V_{L+}| |1 + \Gamma_L e^{-2j\beta l}| = |V_{L+}| |1 + |\Gamma_L| e^{j(\theta_L - 2\beta l)}|$$

At all locations  $l$  for which  $\theta_L - 2\beta l = \pm 2\pi n$ , where  $n$  is an integer, we will have  $\Gamma_l = |\Gamma_L|$  and  $|V_l|$  will be equal to  $V_{\max}$ . Similarly, at all locations for which  $\theta_L - 2\beta l = \pm(2n + 1)\pi$ , we will have  $\Gamma_l = -|\Gamma_L|$  and  $|V_l|$  will be equal to  $V_{\min}$ .

We note that two successive maxima, or two successive minima, are separated by a distance  $\lambda/2$  and a maximum is separated by the next minimum by a distance  $\lambda/4$ , so that  $|l_{\max} - l_{\min}| = \lambda/4$ .

Once such distances  $l_{\max}, l_{\min}$  have been determined, the full reflection coefficient can be constructed from  $\Gamma_L = \Gamma_l e^{2j\beta l}$ , where  $\Gamma_l = \pm |\Gamma_L|$  depending on using a maximum- or minimum-voltage distance  $l$ . From  $\Gamma_L$  and the knowledge of the line impedance  $Z_0$ , the load impedance  $Z_L$  can be computed. Thus, we have:

$$\Gamma_L = |\Gamma_L| e^{j\theta_L} = |\Gamma_L| e^{2j\beta l_{\max}} = -|\Gamma_L| e^{2j\beta l_{\min}} \Rightarrow Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \quad (9.13.1)$$

If  $0 \leq \theta_L \leq \pi$ , the locations for the closest maxima and minima to the load are determined from the conditions:

$$\theta_L - 2\beta l_{\max} = 0, \quad \theta_L - 2\beta l_{\min} = -\pi$$

resulting in the distances:

$$l_{\max} = \frac{\theta_L}{4\pi} \lambda, \quad l_{\min} = \frac{\theta_L + \pi}{4\pi} \lambda, \quad (0 \leq \theta_L \leq \pi) \quad (9.13.2)$$

Similarly, if  $-\pi \leq \theta_L \leq 0$ , we must solve  $\theta_L - 2\beta l_{\max} = -2\pi$  and  $\theta_L - 2\beta l_{\min} = -\pi$ :

$$l_{\max} = \frac{\theta_L + 2\pi}{4\pi} \lambda, \quad l_{\min} = \frac{\theta_L + \pi}{4\pi} \lambda, \quad (-\pi \leq \theta_L \leq 0) \quad (9.13.3)$$

Of course, one wants to solve for  $\theta_L$  in terms of the measured  $l_{\max}$  or  $l_{\min}$ . Using  $l_{\min}$  is more convenient than using  $l_{\max}$  because  $\theta_L$  is given by the same expression in both cases. The lengths  $l_{\max}$ ,  $l_{\min}$  may be assumed to be less than  $\lambda/2$  (if not, we may subtract enough multiples of  $\lambda/2$  until they are.) Expressing  $\theta_L$  in terms of the measured  $l_{\min}$ , we have:

$$\theta_L = \frac{4\pi l_{\min}}{\lambda} - \pi = 2\beta l_{\min} - \pi \quad (9.13.4)$$

Alternatively, we have in terms of  $l_{\max}$ :

$$\theta_L = \begin{cases} \frac{4\pi l_{\max}}{\lambda} = 2\beta l_{\max} & \text{if } 0 \leq l_{\max} \leq \frac{\lambda}{4} \\ \frac{4\pi l_{\max}}{\lambda} - 2\pi = 2\beta l_{\max} - 2\pi & \text{if } \frac{\lambda}{4} \leq l_{\max} \leq \frac{\lambda}{2} \end{cases} \quad (9.13.5)$$

**Example 9.13.1:** A 50-ohm line is connected to an unknown impedance. Voltage measurements along the line reveal that the maximum and minimum voltage values are 1.75 V and 0.25 V. Moreover, the closest distance to the load at which a voltage maximum is observed is  $0.125\lambda$ .

Determine the reflection coefficient  $\Gamma_L$ , the load impedance  $Z_L$ , and the closest distance to the load at which a voltage minimum is observed.

For another load, the same maxima and minima are observed, but now the closest distance to the load at which a minimum is observed is  $0.125\lambda$ . Determine  $\Gamma_L$  and  $Z_L$ .

**Solution:** The SWR is determined to be  $S = V_{\max}/V_{\min} = 1.75/0.25 = 7$ . Then, the magnitude of the reflection coefficient is found to be  $|\Gamma_L| = (S - 1)/(S + 1) = (7 - 1)/(7 + 1) = 0.75$ .

Given that at  $l_{\max} = \lambda/8$  we observe a voltage maximum, we compute the phase from Eq. (9.13.5),  $\theta_L = 2\beta l_{\max} = 4\pi/8 = \pi/2$ . Then, the reflection coefficient will be:

$$\Gamma_L = |\Gamma_L|e^{j\theta_L} = 0.75e^{j\pi/2} = 0.75j$$

It follows that the load impedance will be:

$$Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} = 50 \frac{1 + 0.75j}{1 - 0.75j} = 14 + 48j \Omega$$

The closest voltage minimum will occur at  $l_{\min} = l_{\max} + \lambda/4 = 0.375\lambda = 3\lambda/8$ . Alternatively, we could have determined the phase from Eq. (9.13.4),  $\theta_L = 2\beta l_{\min} - \pi = 4\pi(3/8) - \pi = \pi/2$ . The left graph of Fig. 9.13.1 shows a plot of  $|V_l|$  versus  $l$ .

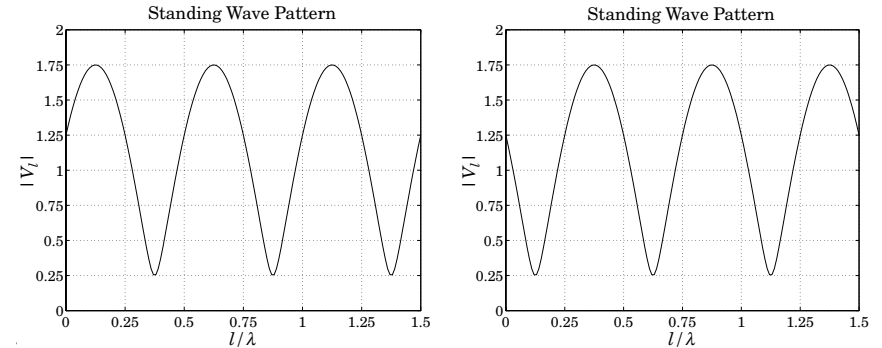


Fig. 9.13.1 Standing wave patterns.

Note the locations of the closest voltage maxima and minima to the load, that is  $\lambda/8$  and  $3\lambda/8$ . In the second case, we are given  $l_{\min} = \lambda/8$ . It follows that  $\theta_L = 2\beta l_{\min} - \pi = \pi/2 - \pi = -\pi/2$ . Alternatively, we may work with  $l_{\max} = l_{\min} + \lambda/4 = 3\lambda/8$ . Because  $l_{\max} > \lambda/4$ , Eq. (9.13.5) will give  $\theta_L = 2\beta l_{\max} - 2\pi = 4\pi(3/8) - 2\pi = -\pi/2$ . The reflection coefficient and load impedance will be:

$$\Gamma_L = |\Gamma_L|e^{j\theta_L} = 0.75e^{-j\pi/2} = -0.75j \Rightarrow Z_L = 14 - 48j \Omega$$

The right graph of Fig. 9.13.1 depicts the standing wave pattern in this case. □

It is interesting also to determine the wave impedances at the locations along the line at which we have voltage maxima or minima, that is, at  $l = l_{\max}$  or  $l_{\min}$ . The answers are expressed in terms of the SWR. Indeed, at  $l = l_{\max}$ , we have  $\Gamma_l = |\Gamma_L|$  which gives:

$$Z_{\max} = Z_0 \frac{1 + \Gamma_l}{1 - \Gamma_l} = Z_0 \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = S Z_0 \quad (9.13.6)$$

Similarly, at  $l = l_{\min}$ , we have  $\Gamma_l = -|\Gamma_L|$  and find:

$$Z_{\min} = Z_0 \frac{1 + \Gamma_l}{1 - \Gamma_l} = Z_0 \frac{1 - |\Gamma_L|}{1 + |\Gamma_L|} = \frac{1}{S} Z_0 \quad (9.13.7)$$

We note that  $Z_{\max}Z_{\min} = Z_0^2$ , as is expected because the points  $l_{\max}$  and  $l_{\min}$  are separated by a quarter-wavelength distance  $\lambda/4$ .

Because at  $l_{\max}$  and  $l_{\min}$  the wave impedances are *real-valued*, these points can be used as convenient locations at which to insert a quarter-wave transformer to match a

line with real  $Z_0$  to a complex load  $Z_L$ . Given  $\theta_L$ , the required locations are determined from Eq. (9.13.2) or (9.13.3). We discuss this matching method later on.

The MATLAB function `lmin.m` calculates the locations  $l_{\min}$  and  $l_{\max}$  from Eqs. (9.13.2) and (9.13.3), and the corresponding impedances  $Z_{\min}$  and  $Z_{\max}$ . It has usage:

```
[lm, Zm] = lmin(ZL, Z0, 'min'); % locations of voltage minima
[lm, Zm] = lmin(ZL, Z0, 'max'); % locations of voltage maxima
```

For a lossless line the power delivered to the load can be measured at any point  $l$  along the line, and in particular, at  $l_{\max}$  and  $l_{\min}$ . Then, Eq. (9.12.7) can be written in the alternative forms:

$$P_L = \frac{1}{2Z_0} (|V_{L+}|^2 - |V_{L-}|^2) = \frac{V_{\max} V_{\min}}{2Z_0} = \frac{V_{\min}^2}{2Z_{\min}} = \frac{V_{\max}^2}{2Z_{\max}} = \frac{V_{\max}^2}{2SZ_0} \quad (9.13.8)$$

The last expression shows that for a given maximum voltage that can be supported along a line, the power transmitted to the load is  $S$  times smaller than it could be if the load were matched.

Conversely, for a given amount  $P_L$  of transmitted power, the maximum voltage will be  $V_{\max} = \sqrt{2SP_L Z_0}$ . One must ensure that for a highly unmatched load,  $V_{\max}$  remain less than the breakdown voltage of the line.

If the line is lossy, measurements of the SWR along its length will give misleading results. Because the reflected power attenuates as it propagates backwards away from the load, the SWR will be smaller at the line input than at the load.

For a lossy line with  $\beta_c = \beta - j\alpha$ , the reflection coefficient at the line input will be:  $\Gamma_d = \Gamma_L e^{-2(\alpha + j\beta)d}$ , which gives for the input SWR:

$$S_d = \frac{1 + |\Gamma_d|}{1 - |\Gamma_d|} = \frac{1 + |\Gamma_L| e^{-2\alpha d}}{1 - |\Gamma_L| e^{-2\alpha d}} = \frac{e^{2\alpha d} + |\Gamma_L|}{e^{2\alpha d} - |\Gamma_L|} = \frac{a + |\Gamma_L|}{a - |\Gamma_L|} \quad (9.13.9)$$

where we expressed it in terms of the matched-line loss of Eq. (9.10.7).

**Example 9.13.2:** For the RG-58 coax cable of Example 9.10.2, we find the SWRs:

$$S_L = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \frac{1 + 0.62}{1 - 0.62} = 4.26, \quad S_d = \frac{1 + |\Gamma_d|}{1 - |\Gamma_d|} = \frac{1 + 0.41}{1 - 0.41} = 2.39$$

If one does not know that the line is lossy, and measures the SWR at the line input, one would think that the load is more matched than it actually is.  $\square$

**Example 9.13.3:** The SWR at the load of a line is 9. If the matched-line loss is 10 dB, what is the SWR at the line input?

**Solution:** We calculate the reflection coefficient at the load:

$$|\Gamma_L| = \frac{S - 1}{S + 1} = \frac{9 - 1}{9 + 1} = 0.8$$

The matched-line loss is  $a = 10^{L_M/10} = 10^{10/10} = 10$ . Thus, the reflection coefficient at the input will be  $|\Gamma_d| = |\Gamma_L|/a = 0.8/10 = 0.08$ . The corresponding SWR will be  $S = (1 + 0.08)/(1 - 0.08) = 1.17$ .  $\square$

**Example 9.13.4:** A 50-ohm line feeds a half-wave dipole antenna with impedance of  $73 + j42.5$  ohms. The line has matched-line loss of 3 dB. What is the total loss of the line? What is the SWR at the load and at the line input?

If the line length is doubled, what is the matched-line loss, the total loss, the input and load SWRs?

**Solution:** The matched-line loss in absolute units is  $a = 10^{3/10} = 2$ . Using the MATLAB functions `z2g` and `swr`, we compute the reflection coefficient at the load and its SWR:

$$|\Gamma_L| = \left| \frac{Z_L - Z_0}{Z_L + Z_0} \right| = \left| \frac{73 + j42.5 - 50}{73 + j42.5 + 50} \right| = \text{abs}(z2g(73 + 42.5j, 50)) = 0.3713$$

The SWR will be  $S = \text{swr}(0.3713) = 2.1814$ . The reflection coefficient at the line input will be  $|\Gamma_d| = |\Gamma_L| e^{-2\alpha d} = |\Gamma_L|/a = 0.1857$ , and its SWR,  $S = \text{swr}(0.1857) = 1.4560$ .

If the line length is doubled, the matched-line loss in dB will double to 6 dB, since it is given by  $L_M = 8.686\alpha d$ . In absolute units, it is  $a = 2^2 = 4$ .

The corresponding reflection coefficient at the line input will be  $|\Gamma_d| = |\Gamma_L|/a = 0.0928$ , and its SWR,  $S = \text{swr}(0.0928) = 1.2047$ .  $\square$

## 9.14 Smith Chart

The relationship between the wave impedance  $Z$  and the corresponding reflection response  $\Gamma$  along a transmission line  $Z_0$  can be stated in terms the normalized impedance  $z = Z/Z_0$  as follows:

$$\Gamma = \frac{z - 1}{z + 1} \Leftrightarrow z = \frac{1 + \Gamma}{1 - \Gamma} \quad (9.14.1)$$

It represents a mapping between the complex impedance  $z$ -plane and the complex reflection coefficient  $\Gamma$ -plane, as shown in Fig. 9.14.1. The mapping is similar to the bilinear transformation mapping in linear system theory between the  $s$ -plane (playing the role of the impedance plane) and the  $z$ -plane of the  $z$ -transform (playing the role of the  $\Gamma$ -plane.)

A complex impedance  $z = r + jx$  with positive resistive part,  $r > 0$ , gets mapped onto a point  $\Gamma$  that lies inside the unit-circle in the  $\Gamma$ -plane, that is, satisfying  $|\Gamma| < 1$ .

An entire resistance line  $z = r$  (a vertical line on the  $z$ -plane) gets mapped onto a circle on the  $\Gamma$ -plane that lies entirely inside the unit-circle, if  $r > 0$ . Similarly, a reactance line  $z = jx$  (a horizontal line on the  $z$ -plane) gets mapped onto a circle on the  $\Gamma$ -plane, a portion of which lies inside the unit-circle.

The Smith chart is a graphical representation of the  $\Gamma$ -plane with a curvilinear grid of constant resistance and constant reactance circles drawn inside the unit-circle. In effect, the Smith chart is a curvilinear graph paper.

Any reflection coefficient point  $\Gamma$  falls at the intersection of a resistance and a reactance circle,  $r, x$ , from which the corresponding impedance can be read off immediately as  $z = r + jx$ . Conversely, given  $z = r + jx$  and finding the intersection of the  $r, x$  circles, the complex point  $\Gamma$  can be located and its value read off in polar or cartesian coordinates.

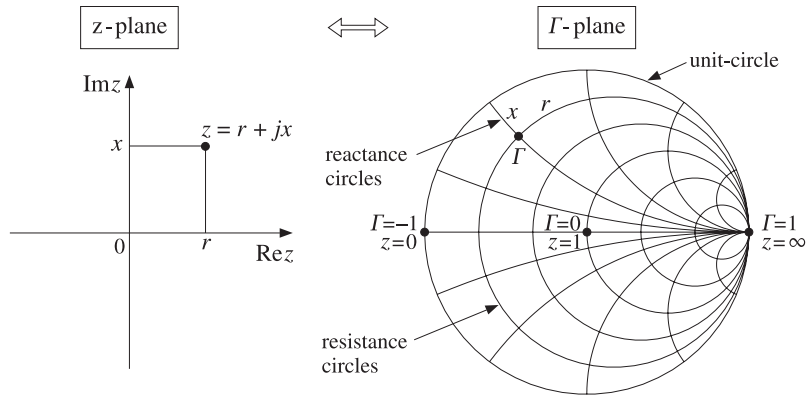


Fig. 9.14.1 Mapping between z-plane and  $\Gamma$ -plane.

To determine the centers and radii of the resistance and reactance circles, we use the result that a circle with center  $C$  and radius  $R$  on the  $\Gamma$ -plane has the following two equivalent representations:

$$|\Gamma|^2 - C^*\Gamma - C\Gamma^* = B \iff |\Gamma - C| = R, \quad \text{where } B = R^2 - |C|^2 \quad (9.14.2)$$

Setting  $z = r + jx$  in Eq. (9.14.1) and extracting the real and imaginary parts, we can write  $r$  and  $x$  in terms of  $\Gamma$ , as follows:

$$r = \text{Re } z = \frac{1 - |\Gamma|^2}{|1 - \Gamma|^2}, \quad x = \text{Im } z = \frac{j(\Gamma^* - \Gamma)}{|1 - \Gamma|^2} \quad (9.14.3)$$

In particular, the expression for the resistive part implies that the condition  $r > 0$  is equivalent to  $|\Gamma| < 1$ . The  $r, x$  circles are obtained by putting Eqs. (9.14.3) in the form of Eq. (9.14.2). We have:

$$r|\Gamma - 1|^2 = 1 - |\Gamma|^2 \implies r(|\Gamma|^2 - \Gamma - \Gamma^* + 1) = 1 - |\Gamma|^2$$

and rearranging terms:

$$|\Gamma|^2 - \frac{r}{r+1}\Gamma - \frac{r}{1+r}\Gamma^* = \frac{1-r}{1+r} \implies \left| \Gamma - \frac{r}{1+r} \right|^2 = \frac{1-r}{1+r} + \frac{r^2}{(1+r)^2} = \left( \frac{1}{1+r} \right)^2$$

Similarly, we have

$$x|\Gamma - 1|^2 = j(\Gamma^* - \Gamma) \implies x(|\Gamma|^2 - \Gamma - \Gamma^* + 1) = j(\Gamma^* - \Gamma)$$

which can be rearranged as:

$$|\Gamma|^2 - \left(1 - \frac{j}{x}\right)\Gamma - \left(1 + \frac{j}{x}\right)\Gamma^* = -1 \implies \left| \Gamma - \left(1 + \frac{j}{x}\right) \right|^2 = -1 + \left(1 + \frac{1}{x^2}\right) = \left(\frac{1}{x}\right)^2$$

To summarize, the constant resistance and reactance circles are:

$$\begin{aligned} \left| \Gamma - \frac{r}{1+r} \right| &= \frac{1}{1+r} && \text{(resistance circles)} \\ \left| \Gamma - \left(1 + \frac{j}{x}\right) \right| &= \frac{1}{|x|} && \text{(reactance circles)} \end{aligned} \quad (9.14.4)$$

The centers of the resistance circles are on the positive half of the real axis on the  $\Gamma$ -plane, lying between  $0 \leq \Gamma \leq 1$ . When  $r = 0$ , the impedance circle is the entire unit-circle with center at  $\Gamma = 0$ . As  $r$  increases, the radii become smaller and the centers move towards  $\Gamma = 1$ . The centers of the reactance circles lie on the tangent of the unit-circle at  $\Gamma = 1$ .

**Example 9.14.1:** Fig. 9.14.2 depicts the resistance and reactance circles for the following values of  $r, x$ :

$$r = [0.2, 0.5, 1, 2, 5], \quad x = [0.2, 0.5, 1, 2, 5]$$

Because the point  $A$  is at the intersection of the  $r = 0.2$  and  $x = 0.5$  circles, the corresponding impedance will be  $z_A = 0.2 + 0.5j$ . We list below the impedances and reflection coefficients at the points  $A, B, C, D, E, S, P, O$ :

	$z_A = 0.2 + 0.5j,$	$\Gamma_A = -0.420 + 0.592j = 0.726 \angle 125.37^\circ$
	$z_B = 0.5 - j,$	$\Gamma_B = 0.077 - 0.615j = 0.620 \angle -82.88^\circ$
	$z_C = 2 - 2j,$	$\Gamma_C = 0.539 - 0.308j = 0.620 \angle -29.74^\circ$
	$z_D = j,$	$\Gamma_D = j = 1 \angle 90^\circ$
	$z_E = -j,$	$\Gamma_E = -j = 1 \angle -90^\circ$
(short circuit)	$z_S = 0,$	$\Gamma_S = -1 = 1 \angle 180^\circ$
(open circuit)	$z_P = \infty,$	$\Gamma_P = 1 = 1 \angle 0^\circ$
(matched)	$z_O = 1,$	$\Gamma_O = 0 = 0 \angle 0^\circ$

The points  $S$  and  $P$  correspond to a short-circuited and an open-circuited impedance. The center of the Smith chart at point  $O$  corresponds to  $z = 1$ , that is, an impedance matched to the line.  $\square$

The Smith chart helps one visualize the wave impedance as one moves away from or towards a load. Assuming a lossless line, the wave impedance and corresponding reflection response at a distance  $l$  from the load are given by:

$$z_l = \frac{z_L + j \tan \beta l}{1 + j z_L \tan \beta l} \iff \Gamma_l = e^{-2j\beta l} \Gamma_L \quad (9.14.5)$$

The magnitude of  $\Gamma_l$  remains constant as  $l$  varies, indeed,  $|\Gamma_l| = |\Gamma_L|$ . On the Smith chart, this represents a circle centered at the origin  $\Gamma = 0$  of radius  $|\Gamma_L|$ . Such circles are called *constant SWR circles* because the SWR is related to the circle radius by

$$S = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

The relative phase angle between  $\Gamma_l$  and  $\Gamma_L$  is negative,  $-2\beta l$ , and therefore, the point  $\Gamma_l$  moves clockwise along the constant SWR circle, as shown in Fig. 9.14.3. Conversely, if  $l$  is decreasing towards the load, the point  $\Gamma_l$  will be moving counter-clockwise.

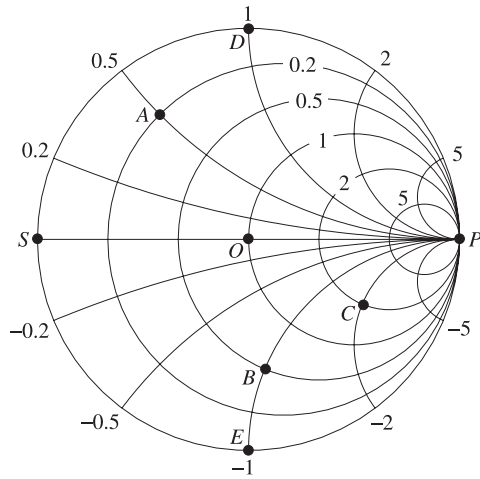


Fig. 9.14.2 Smith chart example.

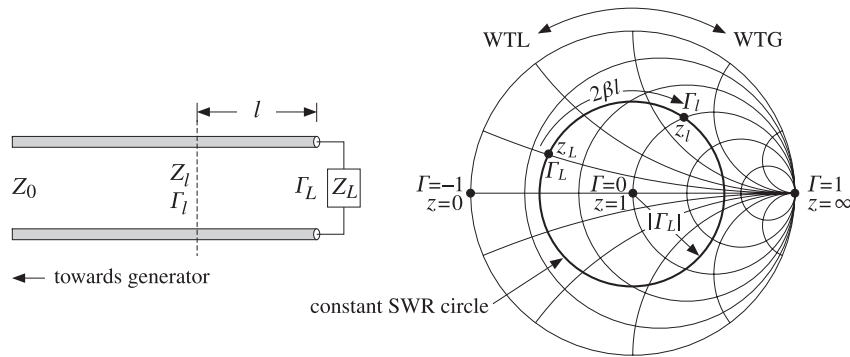


Fig. 9.14.3 Moving towards the generator along a constant SWR circle.

The rotation angle  $\phi_l = 2\beta l$  can be read off in degrees from the outer periphery of the Smith chart. The corresponding length  $l$  can also be read off in units of wavelengths towards the generator (WTG) or wavelengths towards the load (WTL). Moving towards the generator by a distance  $l = \lambda/8$  corresponds to a clockwise rotation by an angle of  $\phi_l = 2(2\pi/8) = \pi/2$ , that is,  $90^\circ$ . Moving by  $l = \lambda/4$  corresponds to a  $180^\circ$  rotation, and by  $l = \lambda/2$ , to a full  $360^\circ$  rotation.

Smith charts provide an intuitive geometrical representation of a load in terms of its reflection coefficient and help one design matching circuits—where matching means moving towards the center of the chart. However, the computational accuracy of the Smith chart is not very high, about 5-10%, because one must visually interpolate between the grid circles of the chart.

Smith charts are used widely to display  $S$ -parameters of microwave amplifiers and help with the design of matching circuits. Some of the tools used in such designs are the stability circles, gain circles, and noise figure circles of an amplifier, which are intuitively represented on a Smith chart. We discuss them in Chap. 12.

Various resources, including a history of the Smith chart and high-quality downloadable charts in Postscript format can be found on the web site [838].

Laursen's Smith chart MATLAB toolbox can be used to draw Smith charts. It is available from the Mathworks web site [849]. Our MATLAB function `smith.m` can be used to draw simple Smith charts.

### 9.15 Time-Domain Response of Transmission Lines

So far we discussed only the sinusoidal response of transmission lines. The response to arbitrary time-domain inputs can be obtained by writing Eq. (9.6.3) in the time domain by replacing  $j\omega \rightarrow \partial/\partial t$ . We will assume a lossless line and set  $R' = G' = 0$ .<sup>†</sup> We obtain then the system of coupled equations:

$$\frac{\partial V}{\partial z} = -L' \frac{\partial I}{\partial t}, \quad \frac{\partial I}{\partial z} = -C' \frac{\partial V}{\partial t} \tag{9.15.1}$$

These are called *telegrapher's equations*. By differentiating again with respect to  $z$ , it is easily verified that  $V$  and  $I$  satisfy the uncoupled one-dimensional wave equations:

$$\frac{\partial^2 V}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = 0, \quad \frac{\partial^2 I}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 I}{\partial t^2} = 0$$

where  $c = 1/\sqrt{L'C'}$ . As in Sec. 2.1, it is better to deal directly with the first-order coupled system (9.15.1). This system can be uncoupled by defining the forward and backward wave components:

$$V_{\pm}(t, z) = \frac{V(t, z) \pm Z_0 I(t, z)}{2}, \quad \text{where } Z_0 = \sqrt{\frac{L'}{C'}} \tag{9.15.2}$$

These satisfy the uncoupled equations:

$$\frac{\partial V_{\pm}}{\partial z} = \mp \frac{1}{c} \frac{\partial V_{\pm}}{\partial t} \tag{9.15.3}$$

with general solutions given in terms of two arbitrary functions  $f(t), g(t)$ :

$$V_+(t, z) = f(t - z/c), \quad V_-(t, z) = g(t + z/c) \tag{9.15.4}$$

These solutions satisfy the basic forward and backward propagation property:

$$\begin{cases} V_+(t, z + \Delta z) = V_+(t - \Delta t, z) \\ V_-(t, z + \Delta z) = V_-(t + \Delta t, z) \end{cases}, \quad \text{where } \Delta t = \frac{\Delta z}{c} \tag{9.15.5}$$

<sup>†</sup>At RF,  $R', G'$  may be small but cannot be assumed to be frequency-independent, for example,  $R'$  depends on the surface impedance  $R_s$ , which grows like  $f^{1/2}$ .

In particular, we have:

$$\begin{aligned} V_+(t, z) &= V_+(t - z/c, 0) \\ V_-(t, z) &= V_-(t + z/c, 0) \end{aligned} \quad (9.15.6)$$

These allow the determination of the line voltages at any point  $z$  along the line from the knowledge of the voltages at  $z = 0$ . Next, we consider a terminated line, shown in Fig. 9.15.1, driven by a generator voltage  $V_G(t)$ , which is typically turned on at  $t = 0$  as indicated by the closing of the switch.

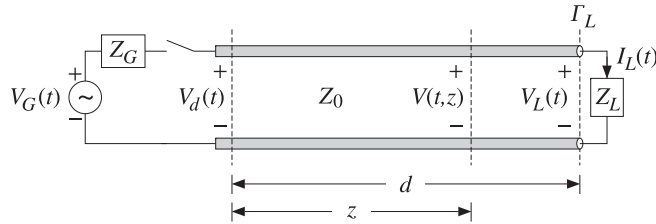


Fig. 9.15.1 Transient response of terminated line.

In general,  $Z_G$  and  $Z_L$  may have inductive or capacitive parts. To begin with, we will assume that they are purely *resistive*. Let the length of the line be  $d$ , so that the one- and two-way travel-time delays will be  $T = d/c$  and  $2T = 2d/c$ .

When the switch closes, an initial waveform is launched forward along the line. When it reaches the load  $T$  seconds later, it gets reflected, picking up a factor of  $\Gamma_L$ , and begins to travel backward. It reaches the generator  $T$  seconds later, or  $2T$  seconds after the initial launch, and gets reflected there traveling forward again, and so on. The total forward- and backward-moving components  $V_{\pm}(t, z)$  include all the multiple reflections.

Before we sum up the multiple reflections, we can express  $V_{\pm}(t, z)$  in terms of the *total* forward-moving component  $V_+(t) \equiv V_+(t, 0)$  at the generator end, with the help of (9.15.6). In fact, we have  $V_+(t, z) = V_+(t - z/c)$ . Applying this at the load end  $z = d$ , we have  $V_L^+(t) = V_+(t, d) = V_+(t - d/c) = V_+(t - T)$ . Because of Ohm's law at the load,  $V_L(t) = Z_L I_L(t)$ , we have for the forward/backward components:

$$V_L^{\pm}(t) = \frac{V_L(t) \pm Z_0 I_L(t)}{2} = \frac{Z_L \pm Z_0}{2} I_L(t) \Rightarrow V_L^-(t) = \frac{Z_L - Z_0}{Z_L + Z_0} V_L^+(t) = \Gamma_L V_+(t - T)$$

Therefore, we find the total voltage at the load end:

$$V_L(t) = V_L^+(t) + V_L^-(t) = (1 + \Gamma_L) V_+(t - T) \quad (9.15.7)$$

Using (9.15.6), the backward component at  $z = 0$  is:

$$\begin{aligned} V_-(t + T) &= V_-(t + d/c, 0) = V_-(t, d) = V_L^-(t) = \Gamma_L V_+(t - T), \quad \text{or,} \\ V_-(t) &= \Gamma_L V_+(t - 2T) \end{aligned}$$

Thus, the total line voltage at the generator end will be:

$$V_d(t) = V_+(t) + V_-(t) = V_+(t) + \Gamma_L V_+(t - 2T) \quad (9.15.8)$$

More generally, the voltage at any point  $z$  along the line will be:

$$V(t, z) = V_+(t, z) + V_-(t, z) = V_+(t - z/c) + \Gamma_L V_+(t + z/c - 2T) \quad (9.15.9)$$

It remains to determine the total forward component  $V_+(t)$  in terms of the multiple reflections of the initially launched wave along the line. We see below that:

$$\begin{aligned} V_+(t) &= \sum_{m=0}^{\infty} (\Gamma_G \Gamma_L)^m V(t - 2mT) \\ &= V(t) + (\Gamma_G \Gamma_L) V(t - 2T) + (\Gamma_G \Gamma_L)^2 V(t - 4T) + \dots \end{aligned} \quad (9.15.10)$$

where  $V(t)$  is the initially launched waveform:

$$V(t) = \frac{Z_0}{Z_G + Z_0} V_G(t) \quad (9.15.11)$$

Thus, initially the transmission line can be replaced by a voltage divider with  $Z_0$  in series with  $Z_L$ . For a right-sided signal  $V(t)$ , such as that generated after closing the switch, the number of terms in (9.15.10) is finite, but growing with time. Indeed, the requirement that the argument of  $V(t - 2mT)$  be non-negative,  $t - 2mT \geq 0$ , may be solved for the limits on  $m$ :

$$0 \leq m \leq M(t), \quad \text{where } M(t) = \text{floor}\left(\frac{t}{2T}\right) \quad (9.15.12)$$

To justify (9.15.10) and (9.15.11), we may start with the single-frequency case discussed in Sec. 9.9 and perform an inverse Fourier transform. Defining the  $z$ -transform variable  $\zeta = e^{j\omega T} = e^{j\beta d}$ ,<sup>†</sup> we may rewrite Eq. (9.9.7) in the form:

$$V_d = V \frac{1 + \Gamma_L \zeta^{-2}}{1 - \Gamma_G \Gamma_L \zeta^{-2}}, \quad Z_0 I_d = V \frac{1 - \Gamma_L \zeta^{-2}}{1 - \Gamma_G \Gamma_L \zeta^{-2}}, \quad \text{where } V = \frac{V_G Z_0}{Z_G + Z_0}$$

The forward and backward waves at  $z = 0$  will be:

$$\begin{aligned} V_+ &= \frac{V_d + Z_0 I_d}{2} = \frac{V}{1 - \Gamma_G \Gamma_L \zeta^{-2}} \\ V_- &= \frac{V_d - Z_0 I_d}{2} = \frac{V \Gamma_L \zeta^{-2}}{1 - \Gamma_G \Gamma_L \zeta^{-2}} = \Gamma_L \zeta^{-2} V_+ \end{aligned} \quad (9.15.13)$$

$$V_d = V_+ + V_- = V_+ + \Gamma_L \zeta^{-2} V_+ \Rightarrow V_d(\omega) = V_+(\omega) + \Gamma_L e^{-2j\omega T} V_+(\omega)$$

where in the last equation we indicated explicitly the dependence on  $\omega$ . Using the delay theorem of Fourier transforms, it follows that the equation for  $V_d(\omega)$  is the Fourier transform of (9.15.8). Similarly, we have at the load end:

<sup>†</sup>We use  $\zeta$  instead of  $z$  to avoid confusion with the position variable  $z$ .

$$V_L = \frac{V_G Z_0}{Z_G + Z_0} \frac{1 + \Gamma_L}{1 - \Gamma_G \Gamma_L \zeta^{-2}} \zeta^{-1} = (1 + \Gamma_L) \zeta^{-1} V_+$$

which is recognized as the Fourier transform of Eq. (9.15.7). Next, we expand  $V_+$  using the geometric series noting that  $|\Gamma_G \Gamma_L \zeta^{-2}| = |\Gamma_G \Gamma_L| < 1$ :

$$V_+ = \frac{V}{1 - \Gamma_G \Gamma_L \zeta^{-2}} = V + (\Gamma_G \Gamma_L) \zeta^{-2} V + (\Gamma_G \Gamma_L)^2 \zeta^{-4} V + \dots \quad (9.15.14)$$

which is equivalent to the Fourier transform of Eq. (9.15.10). The same results can be obtained using a *lattice timing diagram*, shown in Fig. 9.15.2, like that of Fig. 4.6.1.

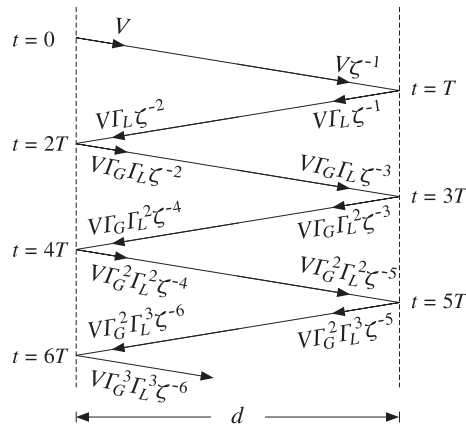


Fig. 9.15.2 Lattice timing diagram.

Each propagation segment introduces a delay factor  $\zeta^{-1}$ , forward or backward, and each reflection at the load and generator ends introduces a factor  $\Gamma_L$  or  $\Gamma_G$ . Summing up all the forward-moving waves at the generator end gives Eq. (9.15.14). Similarly, the summation of the backward terms at the generator, and the summation of the forward and backward terms at the load, give:

$$\begin{aligned} V_- &= V\Gamma_L\zeta^{-2}[1 + (\Gamma_G\Gamma_L)\zeta^{-2} + (\Gamma_G\Gamma_L)^2\zeta^{-4} + \dots] = \Gamma_L\zeta^{-2}V_+ \\ V_L^+ &= V\zeta^{-1}[1 + (\Gamma_G\Gamma_L)\zeta^{-2} + (\Gamma_G\Gamma_L)^2\zeta^{-4} + \dots] = \zeta^{-1}V_+ \\ V_L^- &= \Gamma_L V\zeta^{-1}[1 + (\Gamma_G\Gamma_L)\zeta^{-2} + (\Gamma_G\Gamma_L)^2\zeta^{-4} + \dots] = \Gamma_L\zeta^{-1}V_+ = \Gamma_L V_L^+ \end{aligned}$$

Replacing  $V_+(t)$  in terms of (9.15.10), we obtain from (9.15.7) and (9.15.8):

$$\begin{aligned} V_d(t) &= V(t) + \left(1 + \frac{1}{\Gamma_G}\right) \sum_{m=1}^{\infty} (\Gamma_G \Gamma_L)^m V(t - 2mT) \\ V_L(t) &= (1 + \Gamma_L) \sum_{m=0}^{\infty} (\Gamma_G \Gamma_L)^m V(t - (2m + 1)T) \end{aligned} \quad (9.15.15)$$

The line voltage at an arbitrary location  $z$  along the line, can be determined from (9.15.9). The substitution of the series expansion of  $V_+$  leads to the expression:

$$V(t, z) = \sum_{m=0}^{\infty} (\Gamma_G \Gamma_L)^m V(t - z/c - 2mT) + \Gamma_L \sum_{k=0}^{\infty} (\Gamma_G \Gamma_L)^k V(t + z/c - 2kT - 2T)$$

For a causal input  $V(t)$ , the allowed ranges for the summation indices  $m, k$  are:

$$0 \leq m \leq \text{floor}\left(\frac{t - z/c}{2T}\right), \quad 0 \leq k \leq \text{floor}\left(\frac{t + z/c - 2T}{2T}\right)$$

**Example 9.15.1:** A terminated line has  $Z_0 = 50, Z_G = 450, Z_L = 150 \Omega$ . The corresponding reflection coefficients are calculated to be:  $\Gamma_G = 0.8$  and  $\Gamma_L = 0.5$ . For simplicity, we take  $c = 1, d = 1, T = d/c = 1$ . First, we consider the transient response of the line to a step generator voltage  $V_G(t) = 10 u(t)$ . The initial voltage input to the line will be:  $V(t) = V_G(t) Z_0 / (Z_G + Z_0) = 10 u(t) \cdot 50 / (450 + 50) = u(t)$ . It follows from (9.15.15) that:

$$V_d(t) = u(t) + 2.25 \sum_{m=1}^{\infty} (0.4)^m u(t - 2mT), \quad V_L(t) = 1.5 \sum_{m=1}^{\infty} (0.4)^m u(t - (2m + 1)T)$$

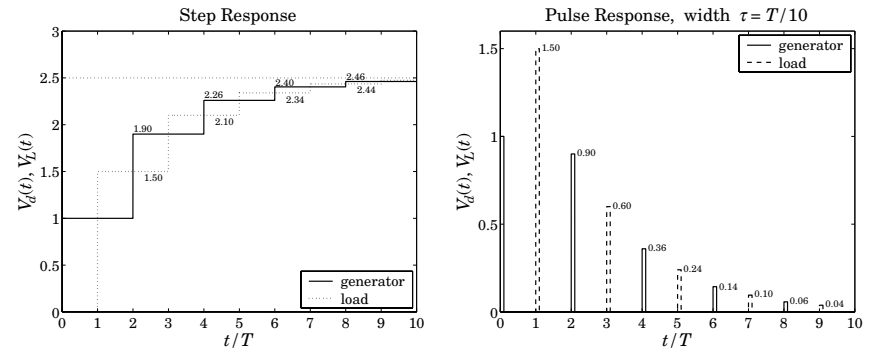


Fig. 9.15.3 Transient step and pulse responses of a terminated line.

These functions are plotted in Fig. 9.15.3. The successive step levels are calculated by:

$V_d(t)$	$V_L(t)$
1	0
$1 + 2.25[0.4^1] = 1.90$	1.5
$1 + 2.25[0.4^1 + 0.4^2] = 2.26$	$1.5[1 + 0.4^1] = 2.10$
$1 + 2.25[0.4^1 + 0.4^2 + 0.4^3] = 2.40$	$1.5([1 + 0.4^1 + 0.4^2]) = 2.34$
$1 + 2.25[0.4^1 + 0.4^2 + 0.4^3 + 0.4^4] = 2.46$	$1.5([1 + 0.4^1 + 0.4^2 + 0.4^3]) = 2.44$

Both  $V_d$  and  $V_L$  converge to the same asymptotic value:

$$1 + 2.25[0.4^1 + 0.4^2 + 0.4^3 + 0.4^4 + \dots] = 1.5[1 + 0.4^1 + 0.4^2 + 0.4^3 + \dots] = \frac{1.5}{1 - 0.4} = 2.5$$

More generally, the asymptotic level for a step input  $V_G(t) = V_G u(t)$  is found to be:

$$V_\infty = V \frac{1 + \Gamma_L}{1 - \Gamma_G \Gamma_L} = \frac{V_G Z_0}{Z_G + Z_0} \frac{1 + \Gamma_L}{1 - \Gamma_G \Gamma_L} = \frac{V_G Z_L}{Z_G + Z_L} \quad (9.15.16)$$

Thus, the line behaves asymptotically like a lumped circuit voltage divider with  $Z_L$  in series with  $Z_G$ . We consider next, the response to a pulse input  $V_G(t) = 10[u(t) - u(t - \tau)]$ , so that  $V(t) = u(t) - u(t - \tau)$ , where  $\tau$  is the pulse duration. Fig. 9.15.3 shows the generator and load line voltages for the case  $\tau = T/10 = 1/10$ . The pulse levels are:

$$[1, 2.25(0.4)^m] = [1.00, 0.90, 0.36, 0.14, 0.06, \dots] \quad (\text{at generator})$$

$$1.5(0.4)^m = [1.50, 0.60, 0.24, 0.10, 0.04, \dots] \quad (\text{at load})$$

The following MATLAB code illustrates the computation of  $V_d(t)$ :

```
d = 1; c=1; T = d/c; tau = T/10; VG = 10;
ZO = 50; ZG = 450; ZL = 150;
V = VG * ZO / (ZG+ZO);
gG = z2g(ZG,ZO); gL = z2g(ZL,ZO);      % reflection coefficients \Gamma_B, \Gamma_L

t = 0 : T/1500 : 10*T;

for i=1:length(t),
    M = floor(t(i)/2/T);
    Vd(i) = V * upulse(t(i), tau);
    if M >= 1,
        m = 1:M;
        Vd(i) = Vd(i) + (1+1/gG)*V*sum((gG*gL).^m .* upulse(t(i)-2*m*T, tau));
    end
end

plot(t, Vd, 'r');
```

where  $upulse(t, \tau)$  generates the unit-pulse function  $u(t) - u(t - \tau)$ . The code can be adapted for any other input function  $V(t)$ .

The MATLAB file `pulsemovie.m` generates a movie of the step or pulse input as it propagates back and forth between generator and load. It plots the voltage  $V(t, z)$  as a function of  $z$  at successive time instants  $t$ . □

Next, we discuss briefly the case of reactive terminations. These are best handled using Laplace transforms. Introducing the  $s$ -domain variable  $s = j\omega$ , we write  $\zeta^{-1} = e^{-j\omega T} = e^{-sT}$ . The terminating impedances, and hence the reflection coefficients, become functions of  $s$ . For example, if the load is a resistor in series with an inductor, we have  $Z_L(s) = R + sL$ . Indicating explicitly the dependence on  $s$ , we have:

$$V_+(s) = \frac{V(s)}{1 - \Gamma_G(s)\Gamma_L(s)e^{-2sT}}, \quad \text{where } V(s) = \frac{V_G(s)Z_0}{Z_G(s) + Z_0} \quad (9.15.17)$$

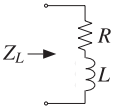
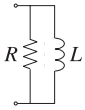
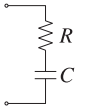
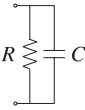
In principle, we may perform an inverse Laplace transform on  $V_+(s)$  to find  $V_+(t)$ . However, this is very tedious and we will illustrate the method only in the case of a matched generator, that is, when  $Z_G = Z_0$ , or,  $\Gamma_G = 0$ . Then,  $V_+(s) = V(s)$ , where

$V(s) = V_G(s)Z_0/2Z_0 = V_G(s)/2$ . The line voltages at the generator and load ends will be from (9.15.13) and (9.15.7):

$$V_d(s) = V(s) + \Gamma_L(s)e^{-2sT}V(s) \quad (9.15.18)$$

$$V_L(s) = [1 + \Gamma_L(s)]e^{-sT}V(s)$$

We consider the four typical cases of series and parallel  $R$ - $L$  and series and parallel  $R$ - $C$  loads. The corresponding  $Z_L(s)$  and  $\Gamma_L(s)$  are shown below, where in all cases  $\Gamma_R = (R - Z_0)/(R + Z_0)$  and the parameter  $a$  gives the effective time constant of the termination,  $\tau = 1/a$ :

series $R$ - $L$	parallel $R$ - $L$	series $R$ - $C$	parallel $R$ - $C$
			
$Z_L(s) = R + sL$	$Z_L(s) = \frac{RsL}{R + sL}$	$Z_L = R + \frac{1}{sC}$	$Z_L(s) = \frac{R}{1 + RCs}$
$\Gamma_L(s) = \frac{s + a\Gamma_R}{s + a}$	$\Gamma_L(s) = \frac{s\Gamma_R - a}{s + a}$	$\Gamma_L(s) = \frac{s\Gamma_R + a}{s + a}$	$\Gamma_L(s) = \frac{-s + a\Gamma_R}{s + a}$
$a = \frac{R + Z_0}{L}$	$a = \frac{Z_0 R}{(R + Z_0)L}$	$a = \frac{1}{(R + Z_0)C}$	$a = \frac{R + Z_0}{RZ_0C}$

We note that in all cases  $\Gamma_L(s)$  has the form:  $\Gamma_L(s) = (b_0s + b_1)/(s + a)$ . Assuming a step-input  $V_G(t) = 2V_0 u(t)$ , we have  $V(t) = V_0 u(t)$ , so that  $V(s) = V_0/s$ . Then,

$$V_d(s) = V_0 \left[ \frac{1}{s} + \Gamma_L(s) \frac{1}{s} e^{-2sT} \right] = V_0 \left[ \frac{1}{s} + \frac{b_0s + b_1}{s(s + a)} e^{-2sT} \right] \quad (9.15.19)$$

Using partial-fraction expansions and the delay theorem of Laplace transforms, we find the inverse Laplace transform:

$$V_d(t) = V_0 u(t) + V_0 \left[ \frac{b_1}{a} + \left( b_0 - \frac{b_1}{a} \right) e^{-a(t-2T)} \right] u(t - 2T) \quad (9.15.20)$$

Applying this result to the four cases, we find:

$$V_d(t) = V_0 u(t) + V_0 [\Gamma_R + (1 - \Gamma_R)e^{-a(t-2T)}] u(t - 2T) \quad (\text{series } R\text{-}L)$$

$$V_d(t) = V_0 u(t) + V_0 [-1 + (1 + \Gamma_R)e^{-a(t-2T)}] u(t - 2T) \quad (\text{parallel } R\text{-}L)$$

$$V_d(t) = V_0 u(t) + V_0 [1 - (1 - \Gamma_R)e^{-a(t-2T)}] u(t - 2T) \quad (\text{series } R\text{-}C)$$

$$V_d(t) = V_0 u(t) + V_0 [\Gamma_R - (1 + \Gamma_R)e^{-a(t-2T)}] u(t - 2T) \quad (\text{parallel } R\text{-}C)$$

In a similar fashion, we determine the load voltage:

$$V_L(t) = V_0 [(1 + \Gamma_R) + (1 - \Gamma_R)e^{-a(t-T)}] u(t - T) \quad (\text{series } R\text{-}L)$$

$$V_L(t) = V_0 (1 + \Gamma_R) e^{-a(t-T)} u(t - T) \quad (\text{parallel } R\text{-}L)$$

$$V_L(t) = V_0 [2 - (1 - \Gamma_R)e^{-a(t-T)}] u(t - T) \quad (\text{series } R\text{-}C)$$

$$V_L(t) = V_0 (1 + \Gamma_R) [1 - e^{-a(t-T)}] u(t - T) \quad (\text{parallel } R\text{-}C)$$



**Example 9.15.2:** We take  $V_0 = 1$ ,  $Z_0 = 50$ ,  $R = 150 \Omega$ , and, as before,  $d = 1$ ,  $c = 1$ ,  $T = 1$ . We find  $\Gamma_R = 0.5$ . Fig. 9.15.4 shows the voltages  $V_d(t)$  and  $V_L(t)$  in the four cases.

In all cases, we adjusted  $L$  and  $C$  such that  $a = 1$ . This gives  $L = 200$  and  $C = 1/200$ , and  $L = 37.5$  and  $C = 1/37.5$ , for the series and parallel cases.

Asymptotically, the series  $R$ - $L$  and the parallel  $R$ - $C$  cases look like a voltage divider  $V_d = V_L = V_G R / (R + Z_0) = 1.5$ , the parallel  $R$ - $L$  case looks like a short-circuited load  $V_d = V_L = 0$ , and the series  $R$ - $C$  looks like an open circuit so that  $V_d = V_L = V_G = 2$ .

Using the expressions for  $V(t, z)$  of Problem 9.30, the MATLAB file `RLCmovi.e.m` makes a movie of the step input as it propagates to and gets reflected from the reactive load. □

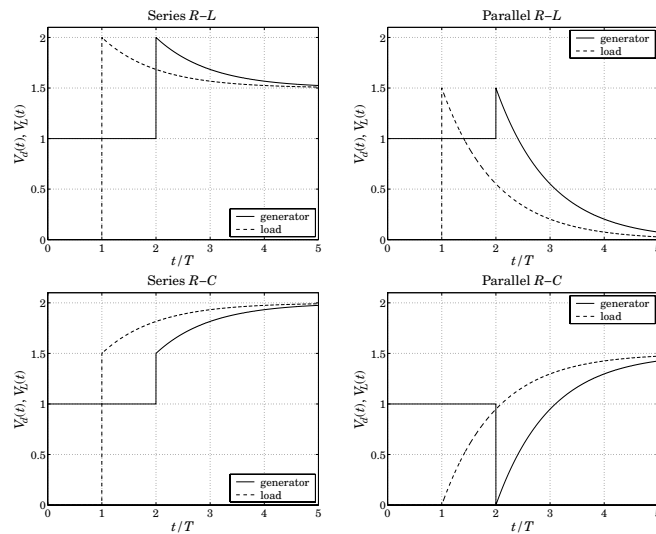


Fig. 9.15.4 Transient response of reactive terminations.

### 9.16 Problems

- 9.1 Design a two-wire line made of two AWG 20-gauge (diameter 0.812 mm) copper wires that has a 300-ohm impedance. Calculate its capacitance per unit length.
- 9.2 For the two-wire line shown in Fig. 9.5.1, show that the tangential component of the electric field vanishes on both cylindrical conductor surfaces. Show that the surface charge and current densities on the positively charged conductor are given in terms of the azimuthal angle  $\phi$  as follows:

$$\rho_s(\phi) = \frac{Q'}{2\pi a} \frac{k^2 - 1}{k^2 - 2k \cos \phi + 1}, \quad J_{sz}(\phi) = \frac{I}{2\pi a} \frac{k^2 - 1}{k^2 - 2k \cos \phi + 1}$$

Show and interpret the following:

$$\int_0^{2\pi} \rho_s(\phi) a d\phi = Q', \quad \int_0^{2\pi} J_{sz}(\phi) a d\phi = I$$

- 9.3 For the two-wire line of the previous problem, show that the power loss per unit length due to ohmic conductor losses is given by:

$$P'_{\text{loss}} = R_s \int_0^{2\pi} |J_{sz}(\phi)|^2 a d\phi = \frac{R_s |I|^2}{2\pi a} \frac{k^2 + 1}{k^2 - 1}$$

From this result, derive Eq. (9.5.13) for  $R'$  and  $\alpha_c$ .

- 9.4 A polyethylene-filled RG-59 coaxial cable has impedance of 75 ohm and velocity factor of 2/3. If the radius of the inner conductor is 0.322 mm, determine the radius of the outer conductor in mm. Determine the capacitance and inductance per unit length. Assuming copper conductors and a loss tangent of  $7 \times 10^{-4}$  for the polyethylene dielectric, calculate the attenuation of the cable in dB/100-ft at 50 MHz and at 1 GHz. Finally, calculate the cutoff frequency of higher propagating modes.
- 9.5 *Computer Experiment: Coaxial Cable Attenuation.* Consider the attenuation data of an RG-8/U cable given in Example 9.4.3.

- a. Reproduce the graph of that Example. Show that with the assumed characteristics of the cable, the total attenuation may be written as a function of frequency in the form, where  $\alpha$  is in dB per 100 ft and  $f$  is in GHz:

$$\alpha(f) = 4.3412 f^{1/2} + 2.9131 f$$

- b. Carry out a least-squares fit of the attenuation data given in the table of that Example by fitting them to a function of the form  $\alpha(f) = A f^{1/2} + B f$ , and determine the fitted coefficients  $A, B$ . This requires that you find  $A, B$  by minimizing the weighted performance index:

$$\mathcal{J} = \sum_i w_i (\alpha_i - A f_i^{1/2} - B f_i)^2 = \min$$

where you may take the weights  $w_i = 1$ . Show that the minimization problem gives rise to a  $2 \times 2$  linear system of equations in the unknowns  $A, B$ , and solve this system with MATLAB.

Plot the resulting function of  $\alpha(f)$  on the same graph as that of part (a). How do the fitted coefficients compare with those of part (a)?

Given the fitted coefficients  $A, B$ , extract from them the estimated values of the loss tangent  $\tan \delta$  and the refractive index  $n$  of the dielectric filling (assuming the cable radii  $a, b$  and conductivity  $\sigma$  are as given.)

- c. Because it appears that the 5-GHz data point is not as accurate as the others, redo part (b) by assigning only 1/2 weight to that point in the least-squares fit. Finally, redo part (b) by assigning zero weight to that point (i.e., not using it in the fit.)

- 9.6 *Computer Experiment: Optimum Coaxial Cables.* Plot the three quantities  $E_a$ ,  $P_T$ , and  $\alpha_c$  given in Eq. (9.4.10) versus the ratio  $b/a$  over the range  $1.5 \leq b/a \leq 4$ . Indicate on the graphs the positions of the optimum ratios that correspond to the minima of  $E_a$  and  $\alpha_c$ , and the maximum of  $P_T$ .

Moreover, write a MATLAB function that solves iteratively (for example, using Newton's method) the equation for minimizing  $\alpha_c$ , that is,  $\ln x = 1 + 1/x$ .

- 9.7 Let  $Z_l = R_l + jX_l$  be the wave impedance on a lossless line at a distance  $l$  from a purely resistive load  $Z_L$ . Derive explicit expressions for  $R_l$  and  $X_l$  in terms of  $Z_L$  and the characteristic impedance  $Z_0$  of the line for the distances  $l = n\lambda/8$ , where  $n = 1, 2, 3, 4, 5, 6, 7, 8$ . Discuss the signs of  $X_l$  (inductive or capacitive) for the two cases  $Z_L > Z_0$  and  $Z_L < Z_0$ . What happens to the above expressions when  $Z_L = Z_0$ ?
- 9.8 A dipole antenna operating in the 30-meter band is connected to a transmitter by a 20-meter long lossless coaxial cable having velocity factor of 0.66 and characteristic impedance of 50 ohm. The wave impedance at the transmitter end of the cable is measured and found to be  $39.9 + 34.2j$  ohm. Determine the input impedance of the antenna.
- 9.9 It is desired to measure the characteristic impedance  $Z_0$  and propagation constant  $\gamma = \alpha + j\beta$  of a lossy line. To this end, a length  $l$  of the line is short-circuited and its input impedance  $Z_{sc}$  is measured. Then, the segment is open-circuited and its input impedance  $Z_{oc}$  is measured. Explain how to extract the two unknown quantities  $Z_0$  and  $\gamma$  from  $Z_{sc}$  and  $Z_{oc}$ .
- 9.10 The wave impedances of a 100-meter long short- and open-circuited segment of a lossy transmission line were measured to be  $Z_{sc} = 68.45 + 128.13j$  ohm and  $Z_{oc} = 4.99 - 16.65j$  ohm at 10 MHz. Using the results of the previous problem, determine the characteristic impedance of the line  $Z_0$ , the attenuation constant  $\alpha$  in dB/100-m, and the velocity factor of the cable noting that the cable length is at least two wavelengths long.
- 9.11 For a lossless line, show the inequality:

$$\frac{1 - |\Gamma_L|}{1 + |\Gamma_L|} \leq \left| \frac{1 + \Gamma_L e^{-2j\beta l}}{1 - \Gamma_L e^{-2j\beta l}} \right| \leq \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

where  $\Gamma_L$  is the load reflection coefficient. Then, show that the magnitude of the wave impedance  $Z_l$  along the line varies between the limits:

$$Z_{\min} \leq |Z_l| \leq Z_{\max}, \quad Z_{\min} = \frac{1}{S} Z_0, \quad Z_{\max} = S Z_0$$

where  $Z_0$  is the characteristic impedance of the line and  $S$ , the voltage SWR.

- 9.12 For a lossless line, show that the current  $I_l$  at a distance  $l$  from a load varies between the limits:

$$I_{\min} \leq |I_l| \leq I_{\max}, \quad \text{where} \quad I_{\min} = \frac{1}{Z_0} V_{\min}, \quad I_{\max} = \frac{1}{Z_0} V_{\max}$$

where  $V_{\min}$  and  $V_{\max}$  are the minimum and maximum voltage along the line. Then, show that the minimum and maximum wave impedances of the previous problem can be written in the alternative forms:

$$Z_{\max} = \frac{V_{\max}}{I_{\min}}, \quad Z_{\min} = \frac{V_{\min}}{I_{\max}}$$

Recall from Sec. 9.13 that  $Z_{\max}$ ,  $Z_{\min}$  correspond to the distances  $l_{\max}$  and  $l_{\min}$ . However, show that  $l_{\min}$  and  $l_{\max}$  correspond to  $l_{\max}$  and  $l_{\min}$ , respectively.

- 9.13 If 500 W of power are delivered to a load by a 50-ohm lossless line and the SWR on the line is 5, determine the maximum voltage  $V_{\max}$  along the line. Determine also the quantities  $V_{\min}$ ,  $I_{\max}$ ,  $I_{\min}$ ,  $Z_{\max}$ , and  $Z_{\min}$ .
- 9.14 A transmitter is connected to an antenna by an 80-ft length of coaxial cable of characteristic impedance of 50 ohm and matched-line loss of 0.6 dB/100-ft. The antenna impedance is  $30 + 40j$  ohm. The transmitter delivers 1 kW of power into the line. Calculate the amount of power delivered to the load and the power lost in the line. Calculate the SWR at the antenna and transmitter ends of the line.

- 9.15 Let  $S_L$  and  $S_d$  be the SWRs at the load and at distance  $d$  from the load on a lossy and mismatched line. Let  $a = e^{2\alpha d}$  be the matched-line loss for the length- $d$  segment. Show that the SWRs are related by:

$$S_d = S_L - \frac{(a-1)(S_L^2-1)}{a(S_L+1)-(S_L-1)} \quad \text{and} \quad S_L = S_d + \frac{(a-1)(S_d^2-1)}{(S_d+1)-a(S_d-1)}$$

Show that  $1 \leq S_d \leq S_L$ . When are the equalities valid? Show also that  $S_d \rightarrow 1$  as  $d \rightarrow \infty$ .

- 9.16 A 100- $\Omega$  lossless transmission line is terminated at an unknown load impedance. The line is operated at a frequency corresponding to a wavelength  $\lambda = 40$  cm. The standing wave ratio along this line is measured to be  $S = 3$ . The distance from the load where there is a voltage minimum is measured to be 5 cm. Based on these two measurements, determine the unknown load impedance.
- 9.17 The wavelength on a 50  $\Omega$  transmission line is 80 cm. Determine the load impedance if the SWR on the line is 3 and the location of the first voltage minimum is 10 cm from the load. At what other distances from the load would one measure a voltage minimum? A voltage maximum?
- 9.18 A 75-ohm line is connected to an unknown load. Voltage measurements along the line reveal that the maximum and minimum voltage values are 6 V and 2 V. It is observed that a voltage maximum occurs at the distance from the load:

$$l = 0.5\lambda - \frac{\lambda}{4\pi} \text{atan}(0.75) = 0.44879\lambda$$

Determine the reflection coefficient  $\Gamma_L$  (in cartesian form) and the load impedance  $Z_L$ .

- 9.19 A load is connected to a generator by a 30-ft long 75-ohm RG-59/U coaxial cable. The SWR is measured at the load and the generator and is found to be equal to 3 and 2, respectively. Determine the attenuation of the cable in dB/ft. Assuming the load is resistive, what are all possible values of the load impedance in ohm?
- 9.20 A lossless 50-ohm line with velocity factor of 0.8 is connected to an unknown load. The operating frequency is 1 GHz. Voltage measurements along the line reveal that the maximum and minimum voltage values are 6 V and 2 V. It is observed that a voltage minimum occurs at a distance of 3 cm from the load. Determine the load reflection coefficient  $\Gamma_L$  and the load impedance  $Z_L$ .
- 9.21 The next four problems are based on Ref. [551]. A lossless transmission line with real characteristic impedance  $Z_0$  is connected to a series RLC circuit.

- a. Show that the corresponding load impedance may be written as a function of frequency in the form (with  $f, f_0$  in Hz):

$$Z_L = R + jRQ \left( \frac{f}{f_0} - \frac{f_0}{f} \right)$$

where  $f_0$  and  $Q$  are the frequency and  $Q$ -factor at resonance. Such a load impedance provides a simplified model for the input impedance of a resonant dipole antenna.

Show that the corresponding SWR  $S_L$  satisfies  $S_L \geq S_0$  for all  $f$ , where  $S_0$  is the SWR at resonance, that is, corresponding to  $Z_L = R$ .

- b. The SWR bandwidth is defined by  $\Delta f = f_2 - f_1$ , where  $f_1, f_2$  are the left and right bandedge frequencies at which the SWR  $S_L$  reaches a certain level, say  $S_L = S_B$ , such that  $S_B > S_0$ . Often the choice  $S_B = 2$  is made. Assuming that  $Z_0 \geq R$ , show that the bandedge frequencies satisfy the conditions:

$$f_1 f_2 = f_0^2, \quad f_1^2 + f_2^2 = 2f_0^2 + f_0^2 \frac{(S_0 + 1)^2 \Gamma_B^2 - (S_0 - 1)^2}{Q^2 (1 - \Gamma_B^2)}, \quad \text{where } \Gamma_B = \frac{S_B - 1}{S_B + 1}$$

- c. Show that the normalized bandwidth is given by:

$$Q \frac{\Delta f}{f_0} = \sqrt{(S_B - S_0)(S_0 - S_B^{-1})} = \sqrt{\frac{4(\Gamma_B^2 - \Gamma_0^2)}{(1 - \Gamma_0)^2 (1 - \Gamma_B^2)}}, \quad \text{with } \Gamma_0 = \frac{S_0 - 1}{S_0 + 1}$$

Show that the left and right bandedge frequencies are given by:

$$f_1 = \sqrt{f_0^2 + \frac{(\Delta f)^2}{4}} - \frac{\Delta f}{2}, \quad f_2 = \sqrt{f_0^2 + \frac{(\Delta f)^2}{4}} + \frac{\Delta f}{2}$$

- d. Show that the maximum bandwidth is realized for a *mismatched* load that has the following optimum SWR at resonance:

$$S_0 = \frac{S_B + S_B^{-1}}{2}, \quad \Gamma_0 = \Gamma_B^2 \Rightarrow Q \frac{\Delta f_{\max}}{f_0} = \frac{S_B^2 - 1}{2S_B} = \frac{2\Gamma_B}{1 - \Gamma_B^2}$$

For example, if  $S_B = 2$ , we have  $\Gamma_B = 1/3$ ,  $S_0 = 1.25$ , and  $\Delta f/f_0 = 0.75/Q$ , whereas for a matched load we have  $S_0 = 1$  and  $\Delta f/f_0 = 0.50/Q$ .

- 9.22 We assume now that the transmission line of the previous problem is lossy and that the RLC load is connected to a generator by a length- $d$  segment of the line. Let  $a = e^{2\alpha d}$  be the matched-line loss. For such lossy line, we may define the bandwidth in terms of the SWR  $S_d$  at the generator end.

Show that the normalized bandwidth is given by the same expression as in the previous problem, but with the replacement  $\Gamma_B \rightarrow \Gamma_{LB}$ , where  $\Gamma_{LB} \equiv a\Gamma_B$ :

$$Q \frac{\Delta f}{f_0} = \sqrt{(S_{LB} - S_0)(S_0 - S_{LB}^{-1})} = \sqrt{\frac{4(\Gamma_{LB}^2 - \Gamma_0^2)}{(1 - \Gamma_0)^2 (1 - \Gamma_{LB}^2)}}, \quad \text{where } S_{LB} = \frac{1 + \Gamma_{LB}}{1 - \Gamma_{LB}}$$

Show that  $\Gamma_{LB}, S_{LB}$  are the quantities  $\Gamma_B, S_B$  referred to the load end of the line. Show that the meaningful range of the bandwidth formula is  $1 \leq S_0 \leq S_{LB}$  in the lossy case, and  $1 \leq S_0 \leq S_B$  for the lossless case. Show that for the same  $S_0$  the bandwidth for the lossy case is always greater than the bandwidth of the lossless case.

Show that this definition of bandwidth makes sense as long as the matched line loss satisfies  $a\Gamma_B < 1$ . Show that the bandwidth vanishes at the  $S_0$  that has  $\Gamma_0 = a\Gamma_B$ . Show that the maximum bandwidth is realized for the optimum  $S_0$ :

$$S_0 = \frac{S_{LB} + S_{LB}^{-1}}{2}, \quad \Gamma_0 = \Gamma_{LB}^2 \Rightarrow Q \frac{\Delta f_{\max}}{f_0} = \frac{S_{LB}^2 - 1}{2S_{LB}} = \frac{2\Gamma_{LB}}{1 - \Gamma_{LB}^2} = \frac{2a\Gamma_B}{1 - a^2\Gamma_B^2}$$

Show that the optimum  $S_0$  is given at the load and generator ends of the line by:

$$S_0 = \frac{1 + a^2\Gamma_B^2}{1 - a^2\Gamma_B^2}, \quad S_{d0} = \frac{1 + a\Gamma_B^2}{1 - a\Gamma_B^2}$$

- 9.23 Assume now that  $Z_0 \leq R$  in the previous problem. Show that the normalized bandwidth is given by:

$$Q \frac{\Delta f}{f_0} = \sqrt{(S_{LB} - S_0^{-1})(S_0^{-1} - S_{LB}^{-1})} = \sqrt{\frac{4(\Gamma_{LB}^2 - \Gamma_0^2)}{(1 + \Gamma_0)^2 (1 - \Gamma_{LB}^2)}}$$

Show that the maximum always occurs at  $S_0 = 1$ . Show that the conditions  $a\Gamma_B < 1$  and  $0 \leq S_0 \leq S_{LB}$  are still required.

Show that, for the same  $S_0$ , the bandwidth of the case  $Z_0 \leq R$  is always smaller than that of the case  $Z_0 \geq R$ .

- 9.24 *Computer Experiment: Antenna Bandwidth.* An 80-meter dipole antenna is resonant at  $f_0 = 3.75$  MHz. Its input impedance is modeled as a series RLC circuit as in Problem 9.21. Its  $Q$ -factor is  $Q = 13$  and its resistance  $R$  at resonance will be varied to achieve various values of the SWR  $S_0$ . The antenna is connected to a transmitter with a length of 75-ohm coaxial cable with matched-line loss of  $a = e^{2\alpha d}$ .

- a. For a lossless line ( $a = 0$  dB), plot the normalized bandwidths  $Q(\Delta f)/f_0$  versus the SWR at the antenna at resonance  $S_0$ . Do two such plots corresponding to SWR bandwidth levels of  $S_B = 2$  and  $S_B = 1.75$ . On the same graphs, add the normalized bandwidth plots for the case of a lossy line with  $a = 2$  dB. Identify on each graph the optimum bandwidth points and the maximum range of  $S_0$  (for convenience, use the same vertical and horizontal scales in all graphs.)

- b. Assume now that  $S_0 = 1.25$ . What are the two possible values of  $R$ ? For these two cases and assuming a lossy line with  $a = 2$  dB, plot the SWR at the antenna end of the line versus frequency in the interval  $3.5 \leq f \leq 4$  MHz. Then, plot the SWRs at the transmitter end of the line. Using common scales on all four graphs, add on each graph the left and right bandedge frequencies corresponding to the two SWR levels of  $S_B = 2$  and  $S_B = 1.75$ . Note the wider bandwidth in the lossy case and for the case having  $Z_0 \geq R$ .

- 9.25 For the special case of a *matched* generator having  $Z_L = Z_0$ , or,  $\Gamma_G = 0$ , show that Eq. (9.15.15) reduces to:

$$V_d(t) = V(t) + \Gamma_L V(t - 2T) \quad \text{and} \quad V_L(t) = (1 + \Gamma_L)V(t - T)$$

- 9.26 A terminated transmission line may be thought of as a *sampled-data linear system*. Show that Eq. (9.15.15) can be written in the convolutional form:

$$V_d(t) = \int_{-\infty}^{\infty} h_d(t')V(t - t') dt', \quad V_L(t) = \int_{-\infty}^{\infty} h_L(t')V(t - t') dt'$$

so that  $V(t)$  may be considered to be the input and  $V_d(t)$  and  $V_L(t)$ , the outputs. Show that the corresponding impulse responses have the sampled-data forms:

$$h_d(t) = \delta(t) + \left(1 + \frac{1}{\Gamma_G}\right) \sum_{m=1}^{\infty} (\Gamma_G \Gamma_L)^m \delta(t - 2mT)$$

$$h_L(t) = (1 + \Gamma_L) \sum_{m=0}^{\infty} (\Gamma_G \Gamma_L)^m \delta(t - (2m + 1)T)$$

What are the corresponding frequency responses? Show that the effective *time constant* of the system may be defined as:

$$\tau = 2T \frac{\ln \epsilon}{\ln |\Gamma_G \Gamma_L|}$$

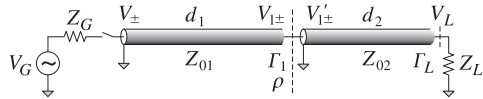
where  $\epsilon$  is a small number, such as  $\epsilon = 10^{-2}$ . Provide an interpretation of  $\tau$ .

9.27 *Computer Experiment: Rise Time and Propagation Effects.* In digital systems where pulses are transmitted along various interconnects, a *rule of thumb* is used according to which if the rise time-constant of a pulse is  $t_r \leq 2.5T$ , where  $T = d/c$  is the propagation delay along the interconnect, then propagation effects must be taken into account. If  $t_r > 5T$ , then a lumped circuit approach may be used.

Consider the transmission line of Example 9.15.1. Using the MATLAB function `upulse.m`, generate four trapezoidal pulses of duration  $t_d = 20T$  and rise times  $t_r = 0, 2.5T, 5T, 10T$ . You may take the fall-times to be equal to the rise-times.

For each pulse, calculate and plot the line voltages  $V_d(t), V_L(t)$  at the generator and load ends for the time period  $0 \leq t \leq 80T$ . Superimpose on these graphs the initial trapezoidal waveform that is launched along the line. Discuss the above rule of thumb in the light of your results.

9.28 Two coaxial transmission lines of lengths  $d_1, d_2$ , impedances  $Z_{01}, Z_{02}$ , and propagation speeds  $c_1, c_2$  are connected in cascade as shown below. Define the one-way travel times and z-transform variables by  $T_1 = d_1/c_1, T_2 = d_2/c_2, \zeta_1 = e^{j\omega T_1}$ , and  $\zeta_2 = e^{j\omega T_2}$ .



Show that the reflection response at the left of the junction is given by:

$$\Gamma_1 = \frac{\rho + \Gamma_L \zeta_2^{-2}}{1 + \rho \Gamma_L \zeta_2^{-2}} = \rho + \frac{\Gamma_L (1 - \rho^2) \zeta_2^{-2}}{1 + \rho \Gamma_L \zeta_2^{-2}}$$

where  $\rho = (Z_{02} - Z_{01}) / (Z_{02} + Z_{01})$  and  $\Gamma_L$  is the load reflection coefficient. Show that the forward and backward voltages at the generator end and to the right of the junction are:

$$V_+ = \frac{V}{1 - \Gamma_G \Gamma_1 \zeta_1^{-2}}, \quad V_- = \Gamma_1 \zeta_1^{-2} V_+, \quad \text{where } V = \frac{V_G Z_{01}}{Z_G + Z_{01}}$$

$$V'_{1+} = \frac{(1 + \rho) \zeta_1^{-1}}{1 + \rho \Gamma_L \zeta_2^{-2}} V_+, \quad V'_{1-} = \frac{(1 + \rho) \Gamma_L \zeta_1^{-1} \zeta_2^{-2}}{1 + \rho \Gamma_L \zeta_2^{-2}} V_+$$

Assume a matched generator, that is, having  $Z_G = Z_{01}$ , or,  $\Gamma_G = 0$ , and a purely resistive load. Show that the time-domain forward and backward transient voltages are given by:

$$V_+(t) = V(t) = \frac{1}{2} V_G(t)$$

$$V_-(t) = \rho V(t - 2T_1) + \Gamma_L (1 - \rho^2) \sum_{m=0}^{\infty} (-\rho \Gamma_L)^m V(t - 2mT_2 - 2T_2 - 2T_1)$$

$$V'_+(t) = (1 + \rho) \sum_{m=0}^{\infty} (-\rho \Gamma_L)^m V(t - 2mT_2 - T_1)$$

$$V'_-(t) = \Gamma_L (1 + \rho) \sum_{m=0}^{\infty} (-\rho \Gamma_L)^m V(t - 2mT_2 - 2T_2 - T_1)$$

Show that the line voltage  $V(t, z)$  is given in terms of the above quantities by:

$$V(t, z) = \begin{cases} V_+(t - z/c_1) + V_-(t + z/c_1), & \text{for } 0 \leq z \leq d_1 \\ V'_{1+}(t - (z - d_1)/c_2) + V'_{1-}(t + (z - d_1)/c_2), & \text{for } d_1 \leq z \leq d_1 + d_2 \end{cases}$$

9.29 *Computer Experiment: Transient Response of Cascaded Lines.* For the previous problem, assume the numerical values  $d_1 = 8, d_2 = 2, c_1 = c_2 = 1, Z_{01} = 50, Z_{02} = 200, Z_G = 50$ , and  $Z_L = 600 \Omega$ .

Plot the line voltage  $V_d(t) = V_+(t) + V_-(t)$  at the generator end for  $0 \leq t \leq 5T_1$ , in the two cases of (a) a step input  $V_G(t) = 3.25 u(t)$ , and (b) a pulse input of width  $\tau = T_1/20$  defined by  $V_G(t) = 3.25[u(t) - u(t - \tau)]$ . You may use the MATLAB functions `ustep.m` and `upulse.m`.

For case (a), explain also the initial and final voltage levels. In both cases, explain the reasons for the time variations of  $V_d(t)$ .

The MATLAB file `pulse2movie.m` generates a movie of the pulse or step signal  $V(t, z)$  as it propagates through this structure.

9.30 Equations (9.15.21) and (9.15.22) represent the line voltages at the generator and load ends of a line terminated by a reactive load. Using inverse Laplace transforms, show that the line voltage at any point z along such a line is given by:

$$V(t, z) = V_0 u(t - z/c) + V_0 [\Gamma_R + (1 - \Gamma_R) e^{-a(t+z/c-2T)}] u(t + z/c - 2T) \quad (\text{series R-L})$$

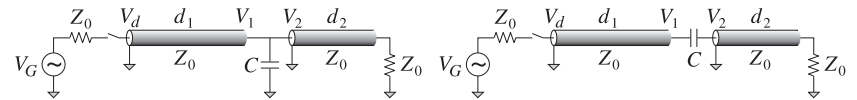
$$V(t, z) = V_0 u(t - z/c) + V_0 [-1 + (1 + \Gamma_R) e^{-a(t+z/c-2T)}] u(t + z/c - 2T) \quad (\text{parallel R-L})$$

$$V(t, z) = V_0 u(t - z/c) + V_0 [1 - (1 - \Gamma_R) e^{-a(t+z/c-2T)}] u(t + z/c - 2T) \quad (\text{series R-C})$$

$$V(t, z) = V_0 u(t - z/c) + V_0 [\Gamma_R - (1 + \Gamma_R) e^{-a(t+z/c-2T)}] u(t + z/c - 2T) \quad (\text{parallel R-C})$$

The MATLAB file `RLCmovie.m` generates a movie of these waves as they propagate to and get reflected from the reactive load.

9.31 *Time-domain reflectometry (TDR)* is used in a number of applications, such as determining fault locations in buried transmission lines, or probing parts of circuit that would otherwise be inaccessible. As a fault-location example, consider a transmission line of impedance  $Z_0$  matched at both the generator and load ends, having a fault at a distance  $d_1$  from the source, or distance  $d_2$  from the load, as shown below.



The fault is shown as a shunt or series capacitor  $C$ . But  $C$  can equally well be replaced by an inductor  $L$ , or a resistor  $R$ . Assuming a unit-step input  $V_G(t) = 2V_0 u(t)$ , show that the TDR voltage  $V_d(t)$  measured at the generator end will be given by:

$$V_d(t) = V_0 u(t) - V_0 e^{-a(t-2T_1)} u(t - 2T_1) \quad (\text{shunt C})$$

$$V_d(t) = V_0 u(t) - V_0 [1 - e^{-a(t-2T_1)}] u(t - 2T_1) \quad (\text{shunt L})$$

$$V_d(t) = V_0 u(t) + V_0 [1 - e^{-a(t-2T_1)}] u(t - 2T_1) \quad (\text{series C})$$

$$V_d(t) = V_0 u(t) + V_0 e^{-a(t-2T_1)} u(t - 2T_1) \quad (\text{series L})$$

$$V_d(t) = V_0 u(t) + V_0 \Gamma_1 u(t - 2T_1) \quad (\text{shunt or series R})$$

where  $T_1 = d_1/c$  is the one-way travel time to the fault. Show that the corresponding time constant  $\tau = 1/a$  is in the four cases:

$$\tau = \frac{Z_0 C}{2}, \quad \tau = 2Z_0 C, \quad \tau = \frac{2L}{Z_0}, \quad \tau = \frac{L}{2Z_0}$$

For a resistive fault, show that  $\Gamma_1 = -Z_0/(2R + Z_0)$ , or,  $\Gamma_1 = R/(2R + Z_0)$ , for a shunt or series  $R$ . Moreover, show that  $\Gamma_1 = (Z_1 - Z_0)/(Z_1 + Z_0)$ , where  $Z_1$  is the parallel (in the shunt- $R$  case) or series combination of  $R$  with  $Z_0$  and give an intuitive explanation of this fact. For a series  $C$ , show that the voltage wave along the two segments is given as follows, and also derive similar expressions for all the other cases:

$$V(t, z) = \begin{cases} V_0 u(t - z/c) + V_0 e^{-a(t+z/c-2T_1)} u(t + z/c - 2T_1), & \text{for } 0 \leq z < d_1 \\ V_0 e^{-a(t-z/c)} u(t - z/c), & \text{for } d_1 < z \leq d_1 + d_2 \end{cases}$$

Make a plot of  $V_d(t)$  for  $0 \leq t \leq 5T_1$ , assuming  $a = 1$  for the  $C$  and  $L$  faults, and  $\Gamma_1 = \mp 1$  corresponding to a shorted shunt or an opened series fault.

The MATLAB file `TDRmovie.m` generates a movie of the step input as it propagates and gets reflected from the fault. The lengths were  $d_1 = 6$ ,  $d_2 = 4$  (in units such that  $c = 1$ ), and the input was  $V_0 = 1$ .