3

Propagation in Birefringent Media

3.1 Linear and Circular Birefringence

In this chapter, we discuss wave propagation in anisotropic media that are linearly or circularly *birefringent*. In such media, uniform plane waves can be decomposed in two *orthogonal* polarization states (linear or circular) that propagate with two different speeds. The two states develop a *phase difference* as they propagate, which alters the total polarization of the wave. Such media are used in the construction of devices for generating different polarizations.

Linearly birefringent materials can be used to change one polarization into another, such as changing linear into circular. Examples are the so-called *uniaxial* crystals, such as calcite, quartz, ice, tourmaline, and sapphire.

Optically active or *chiral* media are circularly birefringent. Examples are sugar solutions, proteins, lipids, nucleic acids, amino acids, DNA, vitamins, hormones, and virtually most other natural substances. In such media, circularly polarized waves go through unchanged, with left- and right-circular polarizations propagating at different speeds. This difference causes linearly polarized waves to have their polarization plane rotate as they propagate—an effect known as *natural optical rotation.*

A similar but not identical effect—the *Faraday rotation*—takes place in *gyroelectric* media, which are ordinary isotropic materials (glass, water, conductors, plasmas) subjected to constant external magnetic fields that break their isotropy. *Gyromagnetic* media, such as ferrites subjected to magnetic fields, also become circularly birefringent.

We discuss all four birefringent cases (linear, chiral, gyroelectric, and gyromagnetic) and the type of constitutive relationships that lead to the corresponding birefringent behavior. We begin by casting Maxwell's equations in different polarization bases.

An arbitrary polarization can be expressed uniquely as a linear combination of two polarizations along two orthogonal directions.† For waves propagating in the z-direction, we may use the two *linear* directions $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$, or the two *circular* ones for right and left polarizations $\{\hat{\mathbf{e}}_+, \hat{\mathbf{e}}_-\}$, where $\hat{\mathbf{e}}_+ = \hat{\mathbf{x}} - j\hat{\mathbf{y}}$ and $\hat{\mathbf{e}}_- = \hat{\mathbf{x}} + j\hat{\mathbf{y}}$.‡ Indeed, we have the following identity relating the linear and circular bases:

†For complex-valued vectors $\mathbf{e}_1, \mathbf{e}_2$, orthogonality is defined with conjugation: $\mathbf{e}_1^* \cdot \mathbf{e}_2 = 0$.

‡Note that $\hat{\mathbf{e}}_\pm$ satisfy: $\hat{\mathbf{e}}_\pm^* \cdot \hat{\mathbf{e}}_\pm = 2$, $\hat{\mathbf{e}}_+^* \cdot \hat{\mathbf{e}}_- = 0$, $\hat{\mathbf{e}}_+ \times \hat{\mathbf{e}}_- = 2j\,\hat{\mathbf{z}}$, and $\hat{\mathbf{z}} \times \hat{\mathbf{e}}_\pm = \pm j\,\hat{\mathbf{e}}_\pm$.

$$\boxed{\boldsymbol{E} = \hat{\mathbf{x}}\,E_x + \hat{\mathbf{y}}\,E_y = \hat{\mathbf{e}}_+\,E_+ + \hat{\mathbf{e}}_-\,E_-}\,, \quad \text{where} \quad E_\pm = \frac{1}{2}(E_x \pm jE_y) \tag{3.1.1}$$

The circular components E_+ and E_- represent right and left polarizations (in the IEEE convention) if the wave is moving in the positive z-direction, but left and right if it is moving in the negative z-direction.

Because the propagation medium is not isotropic, we need to start with the source-free Maxwell's equations before we assume any particular constitutive relationships:

$$\nabla \times \boldsymbol{E} = -j\omega\boldsymbol{B}\,, \quad \nabla \times \boldsymbol{H} = j\omega\boldsymbol{D}\,, \quad \nabla \cdot \boldsymbol{D} = 0\,, \quad \nabla \cdot \boldsymbol{B} = 0 \tag{3.1.2}$$

For a uniform plane wave propagating in the z-direction, we may replace the gradient by $\nabla = \hat{\mathbf{z}}\,\partial_z$. It follows that the curls $\nabla \times \boldsymbol{E} = \hat{\mathbf{z}} \times \partial_z\boldsymbol{E}$ and $\nabla \times \boldsymbol{H} = \hat{\mathbf{z}} \times \partial_z\boldsymbol{H}$ will be transverse to the z-direction. Then, Faraday's and Ampère's laws imply that $D_z = 0$ and $B_z = 0$, and hence both of Gauss' laws are satisfied. Thus, we are left only with:

$$\begin{aligned} \hat{\mathbf{z}} \times \partial_z\boldsymbol{E} &= -j\omega\boldsymbol{B} \\ \hat{\mathbf{z}} \times \partial_z\boldsymbol{H} &= j\omega\boldsymbol{D} \end{aligned} \tag{3.1.3}$$

These equations do not "see" the components E_z, H_z. However, in all the cases that we consider here, the conditions $D_z = B_z = 0$ will imply also that $E_z = H_z = 0$. Thus, all fields are transverse, for example, $\boldsymbol{E} = \hat{\mathbf{x}}\,E_x + \hat{\mathbf{y}}\,E_y = \hat{\mathbf{e}}_+\,E_+ + \hat{\mathbf{e}}_-\,E_-$. Equating x, y components in the two sides of Eq. (3.1.3), we find in the linear basis:

$$\boxed{\begin{aligned} \partial_z E_x &= -j\omega B_y\,, & \partial_z E_y &= j\omega B_x \\ \partial_z H_y &= -j\omega D_x\,, & \partial_z H_x &= j\omega D_y \end{aligned}} \quad \text{(linear basis)} \tag{3.1.4}$$

Using the vector property $\hat{\mathbf{z}} \times \hat{\mathbf{e}}_\pm = \pm j\,\hat{\mathbf{e}}_\pm$ and equating circular components, we obtain the circular-basis version of Eq. (3.1.3) (after canceling some factors of j):

$$\boxed{\begin{aligned} \partial_z E_\pm &= \mp\omega B_\pm \\ \partial_z H_\pm &= \pm\omega D_\pm \end{aligned}} \quad \text{(circular basis)} \tag{3.1.5}$$

3.2 Uniaxial and Biaxial Media

In uniaxial and biaxial homogeneous anisotropic dielectrics, the D–E constitutive relationships are given by the following diagonal forms, where in the biaxial case all diagonal elements of the permittivity matrix are distinct:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_e & 0 & 0 \\ 0 & \epsilon_o & 0 \\ 0 & 0 & \epsilon_o \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{3.2.1}$$

For the uniaxial case, the x-axis is taken to be the *extraordinary* axis with $\epsilon_1 = \epsilon_e$, whereas the y and z axes are *ordinary* axes with permittivities $\epsilon_2 = \epsilon_3 = \epsilon_o$.

The ordinary z-axis was chosen to be the propagation direction in order for the transverse x, y axes to correspond to two different permittivities. In this respect, the uniaxial and biaxial cases are similar, and therefore, we will work with the biaxial case. Setting $D_x = \epsilon_1 E_x$ and $D_y = \epsilon_2 E_y$ in Eq. (3.1.4) and assuming $\boldsymbol{B} = \mu_0 \boldsymbol{H}$, we have:

$$\begin{aligned} \partial_z E_x &= -j\omega\mu_0 H_y\,, \quad & \partial_z E_y &= j\omega\mu_0 H_x \\ \partial_z H_y &= -j\omega\epsilon_1 E_x\,, \quad & \partial_z H_x &= j\omega\epsilon_2 E_y \end{aligned} \tag{3.2.2}$$

Differentiating these once more with respect to z, we obtain the decoupled Helmholtz equations for the x-polarized and y-polarized components:

$$\begin{aligned} \partial_z^2 E_x &= -\omega^2\mu_0\epsilon_1 E_x \\ \partial_z^2 E_y &= -\omega^2\mu_0\epsilon_2 E_y \end{aligned} \tag{3.2.3}$$

The forward-moving solutions are:

$$\begin{aligned} E_x(z) &= Ae^{-jk_1 z}\,, \quad & k_1 &= \omega\sqrt{\mu_0\epsilon_1} = k_0 n_1 \\ E_y(z) &= Be^{-jk_2 z}\,, \quad & k_2 &= \omega\sqrt{\mu_0\epsilon_2} = k_0 n_2 \end{aligned} \tag{3.2.4}$$

where $k_0 = \omega\sqrt{\mu_0\epsilon_0} = \omega/c_0$ is the free-space wavenumber and we defined the refractive indices $n_1 = \sqrt{\epsilon_1/\epsilon_0}$ and $n_2 = \sqrt{\epsilon_2/\epsilon_0}$. Therefore, the total transverse field at $z = 0$ and at distance $z = l$ inside the medium will be:

$$\begin{aligned} \boldsymbol{E}(0) &= \hat{\mathbf{x}}A + \hat{\mathbf{y}}B \\ \boldsymbol{E}(l) &= \hat{\mathbf{x}}Ae^{-jk_1 l} + \hat{\mathbf{y}}Be^{-jk_2 l} = \big[\hat{\mathbf{x}}A + \hat{\mathbf{y}}Be^{j(k_1-k_2)l}\big]e^{-jk_1 l} \end{aligned} \tag{3.2.5}$$

The relative phase $\phi = (k_1 - k_2)l$ between the x- and y-components introduced by the propagation is called *retardance*:

$$\boxed{\phi = (k_1 - k_2)l = (n_1 - n_2)k_0 l = (n_1 - n_2)\frac{2\pi l}{\lambda}} \tag{3.2.6}$$

where λ is the free-space wavelength. Thus, the polarization nature of the field keeps changing as it propagates.

In order to change linear into circular polarization, the wave may be launched into the birefringent medium with a linear polarization having equal x- and y-components. After it propagates a distance l such that $\phi = (n_1 - n_2)k_0 l = \pi/2$, the wave will have changed into left-handed circular polarization:

$$\begin{aligned} \boldsymbol{E}(0) &= A(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \\ \boldsymbol{E}(l) &= A(\hat{\mathbf{x}} + \hat{\mathbf{y}}\,e^{j\phi})e^{-jk_1 l} = A(\hat{\mathbf{x}} + j\,\hat{\mathbf{y}})e^{-jk_1 l} \end{aligned} \tag{3.2.7}$$

Polarization-changing devices that employ this property are called *retarders* and are shown in Fig. 3.2.1. The above example is referred to as a *quarter-wave retarder* because the condition $\phi = \pi/2$ may be written as $(n_1 - n_2)l = \lambda/4$.

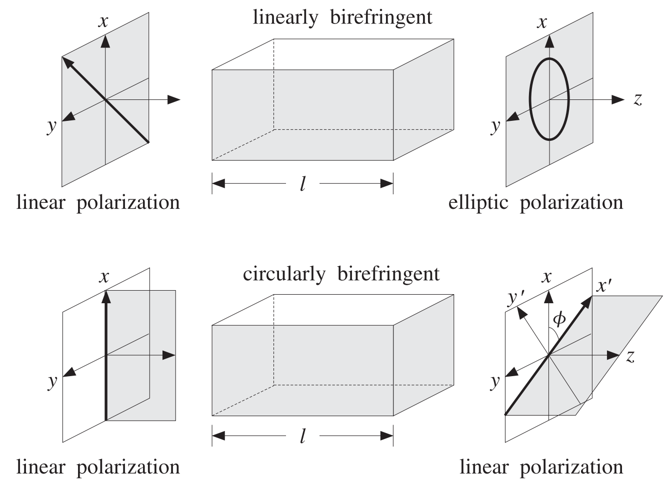

Fig. 3.2.1 Linearly and circularly birefringent retarders.

3.3 Chiral Media

Ever since the first experimental observations of optical activity by Arago and Biot in the early 1800s and Fresnel's explanation that optical rotation is due to circular birefringence, there have been many attempts to explain it at the molecular level. Pasteur was the first to postulate that optical activity is caused by the chirality of molecules.

There exist several versions of constitutive relationships that lead to circular birefringence [285–301]. For single-frequency waves, they are all equivalent to each other. For our purposes, the following so-called Tellegen form is the most convenient [34]:

$$\boxed{\begin{aligned} \boldsymbol{D} &= \epsilon\boldsymbol{E} - j\chi\boldsymbol{H} \\ \boldsymbol{B} &= \mu\boldsymbol{H} + j\chi\boldsymbol{E} \end{aligned}} \qquad \text{(chiral media)} \tag{3.3.1}$$

where χ is a parameter describing the chirality properties of the medium.

It can be shown that the reality (for a lossless medium) and positivity of the energy density function $(\boldsymbol{E}^*\cdot\boldsymbol{D} + \boldsymbol{H}^*\cdot\boldsymbol{B})/2$ requires that the constitutive matrix

$$\begin{bmatrix} \epsilon & -j\chi \\ j\chi & \mu \end{bmatrix}$$

be hermitian and positive definite. This implies that ϵ, μ, χ are real, and furthermore, that $|\chi| < \sqrt{\mu\epsilon}$. Using Eqs. (3.3.1) in Maxwell's equations (3.1.5), we obtain:

$$\begin{aligned} \partial_z E_\pm &= \mp\omega B_\pm = \mp\omega(\mu H_\pm + j\chi E_\pm) \\ \partial_z H_\pm &= \pm\omega D_\pm = \pm\omega(\epsilon E_\pm - j\chi H_\pm) \end{aligned} \tag{3.3.2}$$

Defining $c = 1/\sqrt{\mu\epsilon}$, $\eta = \sqrt{\mu/\epsilon}$, $k = \omega/c = \omega\sqrt{\mu\epsilon}$, and the following real-valued dimensionless parameter $a = c\chi = \chi/\sqrt{\mu\epsilon}$ (so that $|a| < 1$), we may rewrite Eqs. (3.3.2) in the following matrix forms:

$$\frac{\partial}{\partial z}\begin{bmatrix} E_\pm \\ \eta H_\pm \end{bmatrix} = \mp \begin{bmatrix} jka & k \\ -k & jka \end{bmatrix}\begin{bmatrix} E_\pm \\ \eta H_\pm \end{bmatrix} \tag{3.3.3}$$

These matrix equations may be diagonalized by appropriate linear combinations. For example, we define the right-polarized (forward-moving) and left-polarized (backward-moving) waves for the $\{E_+, H_+\}$ case:

$$\begin{aligned} E_{R+} &= \frac{1}{2}[E_+ - j\eta H_+] \\ E_{L+} &= \frac{1}{2}[E_+ + j\eta H_+] \end{aligned} \quad\Leftrightarrow\quad \begin{aligned} E_+ &= E_{R+} + E_{L+} \\ H_+ &= -\frac{1}{j\eta}[E_{R+} - E_{L+}] \end{aligned} \tag{3.3.4}$$

It then follows from Eq. (3.3.3) that $\{E_{R+}, E_{L+}\}$ will satisfy the decoupled equations:

$$\frac{\partial}{\partial z}\begin{bmatrix} E_{R+} \\ E_{L+} \end{bmatrix} = \begin{bmatrix} -jk_+ & 0 \\ 0 & jk_- \end{bmatrix}\begin{bmatrix} E_{R+} \\ E_{L+} \end{bmatrix} \quad\Rightarrow\quad \begin{aligned} E_{R+}(z) &= A_+\, e^{-jk_+z} \\ E_{L+}(z) &= B_+\, e^{jk_-z} \end{aligned} \tag{3.3.5}$$

where k_+, k_- are defined as follows:

$$\boxed{k_\pm = k(1 \pm a) = \omega\big(\sqrt{\mu\epsilon} \pm \chi\big)} \tag{3.3.6}$$

We may also define circular refractive indices by $n_\pm = k_\pm/k_0$, where k_0 is the free-space wavenumber, $k_0 = \omega\sqrt{\mu_0\epsilon_0}$. Setting also $n = k/k_0 = \sqrt{\mu\epsilon}/\sqrt{\mu_0\epsilon_0}$, we have:

$$k_\pm = n_\pm k_0\,, \quad n_\pm = n(1 \pm a) \tag{3.3.7}$$

For the $\{E_-, H_-\}$ circular components, we define the left-polarized (forward-moving) and right-polarized (backward-moving) fields by:

$$\begin{aligned} E_{L-} &= \frac{1}{2}[E_- + j\eta H_-] \\ E_{R-} &= \frac{1}{2}[E_- - j\eta H_-] \end{aligned} \quad\Leftrightarrow\quad \begin{aligned} E_- &= E_{L-} + E_{R-} \\ H_- &= \frac{1}{j\eta}[E_{L-} - E_{R-}] \end{aligned} \tag{3.3.8}$$

Then, $\{E_{L-}, E_{R-}\}$ will satisfy:

$$\frac{\partial}{\partial z}\begin{bmatrix} E_{L-} \\ E_{R-} \end{bmatrix} = \begin{bmatrix} -jk_- & 0 \\ 0 & jk_+ \end{bmatrix}\begin{bmatrix} E_{L-} \\ E_{R-} \end{bmatrix} \quad\Rightarrow\quad \begin{aligned} E_{L-}(z) &= A_-\, e^{-jk_-z} \\ E_{R-}(z) &= B_-\, e^{jk_+z} \end{aligned} \tag{3.3.9}$$

In summary, we obtain the complete circular-basis fields $E_\pm(z)$:

$$\boxed{\begin{aligned} E_+(z) &= E_{R+}(z) + E_{L+}(z) = A_+\, e^{-jk_+z} + B_+\, e^{jk_-z} \\ E_-(z) &= E_{L-}(z) + E_{R-}(z) = A_-\, e^{-jk_-z} + B_-\, e^{jk_+z} \end{aligned}} \tag{3.3.10}$$

Thus, the $E_+(z)$ circular component propagates forward with wavenumber k_+ and backward with k_-, and the reverse is true of the $E_-(z)$ component. The forward-moving component of E_+ and the backward-moving component of E_-, that is, E_{R+} and E_{R-}, are

both right-polarized and both propagate with the same wavenumber k_+. Similarly, the left-polarized waves E_{L+} and E_{L-} both propagate with k_-.

Thus, a wave of given circular polarization (left or right) propagates with the same wavenumber regardless of its direction of propagation. This is a characteristic difference of chiral versus gyrotropic media in external magnetic fields.

Consider, next, the effect of natural rotation. We start with a linearly polarized field at $z = 0$ and decompose it into its circular components:

$$\mathbf{E}(0) = \hat{\mathbf{x}}\,A_x + \hat{\mathbf{y}}\,A_y = \hat{\mathbf{e}}_+ A_+ + \hat{\mathbf{e}}_- A_-\,, \quad \text{with} \quad A_\pm = \frac{1}{2}(A_x \pm jA_y)$$

where A_x, A_y must be real for linear polarization. Propagating the circular components forward by a distance l according to Eq. (3.3.10), we find:

$$\begin{aligned} \mathbf{E}(l) &= \hat{\mathbf{e}}_+ A_+\, e^{-jk_+l} + \hat{\mathbf{e}}_- A_-\, e^{-jk_-l} \\ &= \big[\hat{\mathbf{e}}_+ A_+ e^{-j(k_+-k_-)l/2} + \hat{\mathbf{e}}_- A_- e^{j(k_+-k_-)l/2}\big] e^{-j(k_++k_-)l/2} \\ &= \big[\hat{\mathbf{e}}_+ A_+ e^{-j\phi} + \hat{\mathbf{e}}_- A_- e^{j\phi}\big] e^{-j(k_++k_-)l/2} \end{aligned} \tag{3.3.11}$$

where we defined the angle of rotation:

$$\boxed{\phi = \frac{1}{2}(k_+ - k_-)l = akl} \qquad \text{(natural rotation)} \tag{3.3.12}$$

Going back to the linear basis, we find:

$$\begin{aligned} \hat{\mathbf{e}}_+ A_+ e^{-j\phi} + \hat{\mathbf{e}}_- A_- e^{j\phi} &= (\hat{\mathbf{x}} - j\hat{\mathbf{y}})\frac{1}{2}(A_x + jA_y)e^{-j\phi} + (\hat{\mathbf{x}} + j\hat{\mathbf{y}})\frac{1}{2}(A_x - jA_y)e^{j\phi} \\ &= [\hat{\mathbf{x}}\cos\phi - \hat{\mathbf{y}}\sin\phi]A_x + [\hat{\mathbf{y}}\cos\phi + \hat{\mathbf{x}}\sin\phi]A_y \\ &= \hat{\mathbf{x}}' A_x + \hat{\mathbf{y}}' A_y \end{aligned}$$

Therefore, at $z = 0$ and $z = l$, we have:

$$\begin{aligned} \mathbf{E}(0) &= \big[\hat{\mathbf{x}}\,A_x + \hat{\mathbf{y}}\,A_y\big] \\ \mathbf{E}(l) &= \big[\hat{\mathbf{x}}' A_x + \hat{\mathbf{y}}' A_y\big] e^{-j(k_++k_-)l/2} \end{aligned} \tag{3.3.13}$$

The new unit vectors $\hat{\mathbf{x}}' = \hat{\mathbf{x}}\cos\phi - \hat{\mathbf{y}}\sin\phi$ and $\hat{\mathbf{y}}' = \hat{\mathbf{y}}\cos\phi + \hat{\mathbf{x}}\sin\phi$ are recognized as the unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ rotated clockwise (if $\phi > 0$) by the angle ϕ, as shown in Fig. 3.2.1 (for the case $A_x \neq 0$, $A_y = 0$.) Thus, the wave remains linearly polarized, but its polarization plane rotates as it propagates.

If the propagation is in the negative z-direction, then as follows from Eq. (3.3.10), the roles of k_+ and k_- are interchanged so that the rotation angle becomes $\phi = (k_- - k_+)l/2$, which is the negative of that of Eq. (3.3.12).

If a linearly polarized wave travels forward by a distance l, gets reflected, and travels back to the starting point, the total angle of rotation will be zero. By contrast, in the Faraday rotation case, the angle keeps increasing so that it doubles after a round trip (see Problem 3.10.)

3.4 Gyrotropic Media

Gyrotropic[†] media are isotropic media in the presence of constant external magnetic fields. A gyroelectric medium (at frequency ω) has constitutive relationships:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & j\epsilon_2 & 0 \\ -j\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}, \quad \boldsymbol{B} = \mu \boldsymbol{H} \tag{3.4.1}$$

For a lossless medium, the positivity of the energy density function requires that the permittivity matrix be hermitian and positive-definite, which implies that $\epsilon_1, \epsilon_2, \epsilon_3$ are real, and moreover, $\epsilon_1 > 0$, $|\epsilon_2| \le \epsilon_1$, and $\epsilon_3 > 0$. The quantity ϵ_2 is proportional to the external magnetic field and reverses sign with the direction of that field.

A gyromagnetic medium, such as a ferrite in the presence of a magnetic field, has similar constitutive relationships, but with the roles of $\boldsymbol{D}$ and $\boldsymbol{H}$ interchanged:

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \mu_1 & j\mu_2 & 0 \\ -j\mu_2 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}, \quad \boldsymbol{D} = \epsilon \boldsymbol{E} \tag{3.4.2}$$

where again $\mu_1 > 0$, $|\mu_2| \le \mu_1$, and $\mu_3 > 0$ for a lossless medium.

In the circular basis of Eq. (3.1.1), the above gyrotropic constitutive relationships take the simplified forms:

$$\begin{aligned} D_\pm &= (\epsilon_1 \pm \epsilon_2) E_\pm\,, \quad B_\pm = \mu H_\pm\,, \quad \text{(gyroelectric)} \\ B_\pm &= (\mu_1 \pm \mu_2) H_\pm\,, \quad D_\pm = \epsilon E_\pm\,, \quad \text{(gyromagnetic)} \end{aligned} \tag{3.4.3}$$

where we ignored the z-components, which are zero for a uniform plane wave propagating in the z-direction. For example,

$$D_x \pm jD_y = (\epsilon_1 E_x + j\epsilon_2 E_y) \pm j(\epsilon_1 E_y - j\epsilon_2 E_x) = (\epsilon_1 \pm \epsilon_2)(E_x \pm jE_y)$$

Next, we solve Eqs. (3.1.5) for the forward and backward circular-basis waves. Considering the gyroelectric case first, we define the following quantities:

$$\boxed{\epsilon_\pm = \epsilon_1 \pm \epsilon_2\,, \quad k_\pm = \omega\sqrt{\mu\epsilon_\pm}\,, \quad \eta_\pm = \sqrt{\frac{\mu}{\epsilon_\pm}}} \tag{3.4.4}$$

Using these definitions and the constitutive relations $D_\pm = \epsilon_\pm E_\pm$, Eqs. (3.1.5) may be rearranged into the following matrix forms:

$$\frac{\partial}{\partial z} \begin{bmatrix} E_\pm \\ \eta_\pm H_\pm \end{bmatrix} = \begin{bmatrix} 0 & \mp k_\pm \\ \pm k_\pm & 0 \end{bmatrix} \begin{bmatrix} E_\pm \\ \eta_\pm H_\pm \end{bmatrix} \tag{3.4.5}$$

These may be decoupled by defining forward- and backward-moving fields as in Eqs. (3.3.4) and (3.3.8), but using the corresponding circular impedances $\eta_\pm$:

$$\begin{aligned} E_{R+} &= \frac{1}{2}[E_+ - j\eta_+ H_+] \qquad E_{L-} = \frac{1}{2}[E_- + j\eta_- H_-] \\ E_{L+} &= \frac{1}{2}[E_+ + j\eta_+ H_+] \qquad E_{R-} = \frac{1}{2}[E_- - j\eta_- H_-] \end{aligned} \tag{3.4.6}$$

[†]The term "gyrotropic" is sometimes also used to mean "optically active."

These satisfy the decoupled equations:

$$\begin{aligned} \frac{\partial}{\partial z} \begin{bmatrix} E_{R+} \\ E_{L+} \end{bmatrix} &= \begin{bmatrix} -jk_+ & 0 \\ 0 & jk_+ \end{bmatrix} \begin{bmatrix} E_{R+} \\ E_{L+} \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} E_{R+}(z) &= A_+ e^{-jk_+ z} \\ E_{L+}(z) &= B_+ e^{jk_+ z} \end{aligned} \\ \frac{\partial}{\partial z} \begin{bmatrix} E_{L-} \\ E_{R-} \end{bmatrix} &= \begin{bmatrix} -jk_- & 0 \\ 0 & jk_- \end{bmatrix} \begin{bmatrix} E_{L-} \\ E_{R-} \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} E_{L-}(z) &= A_- e^{-jk_- z} \\ E_{R-}(z) &= B_- e^{jk_- z} \end{aligned} \end{aligned} \tag{3.4.7}$$

Thus, the complete circular-basis fields $E_\pm(z)$ are:

$$\boxed{\begin{aligned} E_+(z) &= E_{R+}(z) + E_{L+}(z) = A_+ e^{-jk_+ z} + B_+ e^{jk_+ z} \\ E_-(z) &= E_{L-}(z) + E_{R-}(z) = A_- e^{-jk_- z} + B_- e^{jk_- z} \end{aligned}} \tag{3.4.8}$$

Now, the $E_+(z)$ circular component propagates forward and backward with the *same* wavenumber k_+, while $E_-(z)$ propagates with k_-. Eq. (3.3.13) and the steps leading to it remain valid here. The rotation of the polarization plane is referred to as the *Faraday rotation*. If the propagation is in the negative z-direction, then the roles of k_+ and k_- remain unchanged so that the rotation angle is still the same as that of Eq. (3.3.12).

If a linearly polarized wave travels forward by a distance l, gets reflected, and travels back to the starting point, the total angle of rotation will be double that of the single trip, that is, $2\phi = (k_+ - k_-)l$.

Problems 1.9 and 3.12 discuss simple models of gyroelectric behavior for conductors and plasmas in the presence of an external magnetic field. Problem 3.14 develops the Appleton-Hartree formulas for plane waves propagating in plasmas, such as the ionosphere [302–306].

The gyromagnetic case is essentially identical to the gyroelectric one. Eqs. (3.4.5) to (3.4.8) remain the same, but with circular wavenumbers and impedances defined by:

$$\boxed{\mu_\pm = \mu_1 \pm \mu_2\,, \quad k_\pm = \omega\sqrt{\epsilon\mu_\pm}\,, \quad \eta_\pm = \sqrt{\frac{\mu_\pm}{\epsilon}}} \tag{3.4.9}$$

Problem 3.13 discusses a model for magnetic resonance exhibiting gyromagnetic behavior. Magnetic resonance has many applications—from NMR imaging to ferrite microwave devices [307–318]. Historical overviews may be found in [316,318].

3.5 Linear and Circular Dichroism

Dichroic polarizers, such as polaroids, are linearly birefringent materials that have widely different attenuation coefficients along the two polarization directions. For a lossy material, the field solutions given in Eq. (3.2.4) are modified as follows:

$$\begin{aligned} E_x(z) &= Ae^{-jk_1 z} = Ae^{-\alpha_1 z} e^{-j\beta_1 z}\,, \qquad k_1 = \omega\sqrt{\mu\epsilon_1} = \beta_1 - j\alpha_1 \\ E_y(z) &= Be^{-jk_2 z} = Be^{-\alpha_2 z} e^{-j\beta_1 2}\,, \qquad k_2 = \omega\sqrt{\mu\epsilon_2} = \beta_2 - j\alpha_2 \end{aligned} \tag{3.5.1}$$

where α_1, α_2 are the attenuation coefficients. Passing through a length l of such a material, the initial and output polarizations will be as follows:

$$
\begin{aligned}
E(0) &= \hat{\mathbf{x}}\,A + \hat{\mathbf{y}}\,B \\
E(l) &= \hat{\mathbf{x}}\,Ae^{-jk_1 l} + \hat{\mathbf{y}}\,Be^{-jk_2 l} = \big(\hat{\mathbf{x}}\,Ae^{-\alpha_1 l} + \hat{\mathbf{y}}\,Be^{-\alpha_2 l}e^{j\phi}\big)\,e^{-j\beta_1 l}
\end{aligned}
\tag{3.5.2}
$$

In addition to the phase change $\phi = (\beta_1 - \beta_2)l$, the field amplitudes have attenuated by the unequal factors $a_1 = e^{-\alpha_1 l}$ and $a_2 = e^{-\alpha_2 l}$. The resulting polarization will be elliptic with unequal semi-axes. If $\alpha_2 \gg \alpha_1$, then $a_2 \ll a_1$ and the y-component can be ignored in favor of the x-component.

This is the basic principle by which a polaroid material lets through only a preferred linear polarization. An ideal linear polarizer would have $a_1 = 1$ and $a_2 = 0$, corresponding to $\alpha_1 = 0$ and $\alpha_2 = \infty$. Typical values of the attenuations for commercially available polaroids are of the order of $a_1 = 0.9$ and $a_2 = 10^{-2}$, or 0.9 dB and 40 dB, respectively.

Chiral media may exhibit *circular dichroism* [287,300], in which the circular wavenumbers become complex, $k_\pm = \beta_\pm - j\alpha_\pm$. Eq. (3.3.11) reads now:

$$
\begin{aligned}
E(l) &= \hat{\mathbf{e}}_+ A_+\,e^{-jk_+ l} + \hat{\mathbf{e}}_- A_-\,e^{-jk_- l} \\
&= \big[\hat{\mathbf{e}}_+ A_+ e^{-j(k_+-k_-)l/2} + \hat{\mathbf{e}}_- A_- e^{j(k_+-k_-)l/2}\big]e^{-j(k_++k_-)l/2} \\
&= \big[\hat{\mathbf{e}}_+ A_+ e^{-\psi-j\phi} + \hat{\mathbf{e}}_- A_- e^{\psi+j\phi}\big]e^{-j(k_++k_-)l/2}
\end{aligned}
\tag{3.5.3}
$$

where we defined the complex rotation angle:

$$
\phi - j\psi = \frac{1}{2}(k_+ - k_-)l = \frac{1}{2}(\beta_+ - \beta_-)l - j\,\frac{1}{2}(\alpha_+ - \alpha_-)l
\tag{3.5.4}
$$

Going back to the linear basis as in Eq. (3.3.13), we obtain:

$$
\begin{aligned}
E(0) &= \big[\hat{\mathbf{x}}\,A_x + \hat{\mathbf{y}}\,A_y\big] \\
E(l) &= \big[\hat{\mathbf{x}}'A'_x + \hat{\mathbf{y}}'A'_y\big]e^{-j(k_++k_-)l/2}
\end{aligned}
\tag{3.5.5}
$$

where $\{\hat{\mathbf{x}}', \hat{\mathbf{y}}'\}$ are the same rotated (by ϕ) unit vectors of Eq. (3.3.13), and

$$
\begin{aligned}
A'_x &= A_x \cosh\psi - jA_y \sinh\psi \\
A'_y &= A_y \cosh\psi + jA_x \sinh\psi
\end{aligned}
\tag{3.5.6}
$$

Because the amplitudes A'_x, A'_y are now complex-valued, the resulting polarization will be elliptical.

3.6 *Oblique Propagation in Birefringent Media*

Here, we discuss TE and TM waves propagating in oblique directions in linearly birefringent media. We will use these results in Chap. 7 to discuss reflection and refraction in such media, and to characterize the properties of birefringent multilayer structures.

Applications include the recently manufactured (by 3M, Inc.) multilayer birefringent polymer mirrors that have remarkable and unusual optical properties, collectively referred to as *giant birefringent optics* (GBO) [264].

Oblique propagation in chiral and gyrotropic media is discussed in the problems. Further discussions of wave propagation in anisotropic media may be found in [31–33].

We recall from Sec. 2.9 that a uniform plane wave propagating in a lossless isotropic dielectric in the direction of a wave vector $\boldsymbol{k}$ is given by:

$$
\boldsymbol{E}(\mathbf{r}) = \boldsymbol{E}\,e^{-j\,\boldsymbol{k}\cdot\mathbf{r}}\,, \quad \boldsymbol{H}(\mathbf{r}) = \boldsymbol{H}\,e^{-j\,\boldsymbol{k}\cdot\mathbf{r}}\,, \quad \text{with}\quad \hat{\mathbf{k}}\cdot\boldsymbol{E} = 0\,, \quad \boldsymbol{H} = \frac{n}{\eta_0}\,\hat{\mathbf{k}}\times\boldsymbol{E}
\tag{3.6.1}
$$

where n is the refractive index of the medium $n = \sqrt{\epsilon/\epsilon_0}$, η_0 the free-space impedance, and $\hat{\mathbf{k}}$ the unit-vector in the direction of $\boldsymbol{k}$, so that,

$$
\boldsymbol{k} = k\,\hat{\mathbf{k}}\,, \quad k = |\boldsymbol{k}| = \omega\sqrt{\mu_0\epsilon} = nk_0\,, \quad k_0 = \frac{\omega}{c_0} = \omega\sqrt{\mu_0\epsilon_0}
\tag{3.6.2}
$$

and k_0 is the free-space wavenumber. Thus, $\boldsymbol{E}, \boldsymbol{H}, \hat{\mathbf{k}}$ form a right-handed system.

In particular, following the notation of Fig. 2.9.1, if $\boldsymbol{k}$ is chosen to lie in the xz plane at an angle θ from the z-axis, that is, $\hat{\mathbf{k}} = \hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$, then there will be two independent polarization solutions: TM, parallel, or p-polarization, and TE, perpendicular, or s-polarization, with fields given by

$$
\begin{aligned}
&\text{(TM or p-polarization):} \qquad \boldsymbol{E} = E_0\,(\hat{\mathbf{x}}\cos\theta - \hat{\mathbf{z}}\sin\theta)\,, \quad \boldsymbol{H} = \frac{n}{\eta_0}\,E_0\,\hat{\mathbf{y}} \\
&\text{(TE or s-polarization):} \qquad \boldsymbol{E} = E_0\,\hat{\mathbf{y}}\,, \quad \boldsymbol{H} = \frac{n}{\eta_0}\,E_0\,(-\hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta)
\end{aligned}
\tag{3.6.3}
$$

where, in both the TE and TM cases, the propagation phase factor $e^{-j\,\boldsymbol{k}\cdot\mathbf{r}}$ is:

$$
e^{-j\,\boldsymbol{k}\cdot\mathbf{r}} = e^{-j(k_z z + k_x x)} = e^{-jk_0 n(z\cos\theta + x\sin\theta)}
\tag{3.6.4}
$$

The designation as parallel or perpendicular is completely arbitrary here and is taken with respect to the xz plane. In the reflection and refraction problems discussed in Chap. 6, the dielectric interface is taken to be the xy plane and the xz plane becomes the plane of incidence.

In a birefringent medium, the propagation of a uniform plane wave with arbitrary wave vector $\boldsymbol{k}$ is much more difficult to describe. For example, the direction of the Poynting vector is not towards $\boldsymbol{k}$, the electric field $\boldsymbol{E}$ is not orthogonal to $\boldsymbol{k}$, the simple dispersion relationship $k = n\omega/c_0$ is not valid, and so on.

In the previous section, we considered the special case of propagation along an ordinary optic axis in a birefringent medium. Here, we discuss the somewhat more general case in which the xyz coordinate axes coincide with the principal dielectric axes (so that the permittivity tensor is diagonal,) and we take the wave vector $\boldsymbol{k}$ to lie in the xz plane at an angle θ from the z-axis. The geometry is depicted in Fig. 3.6.1.

Although this case is still not the most general one with a completely arbitrary direction for $\boldsymbol{k}$, it does contain most of the essential features of propagation in birefringent media. The 3M multilayer films mentioned above have similar orientations for their optic axes [264].

The constitutive relations are assumed to be $\boldsymbol{B} = \mu_0\boldsymbol{H}$ and a diagonal permittivity tensor for $\boldsymbol{D}$. Let $\epsilon_1, \epsilon_2, \epsilon_3$ be the permittivity values along the three principal axes and define the corresponding refractive indices $n_i = \sqrt{\epsilon_i/\epsilon_0}$, $i = 1, 2, 3$. Then, the $\boldsymbol{D}$-$\boldsymbol{E}$ relationship becomes:

$$
\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \epsilon_0\begin{bmatrix} n_1^2 & 0 & 0 \\ 0 & n_2^2 & 0 \\ 0 & 0 & n_3^2 \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}
\tag{3.6.5}
$$

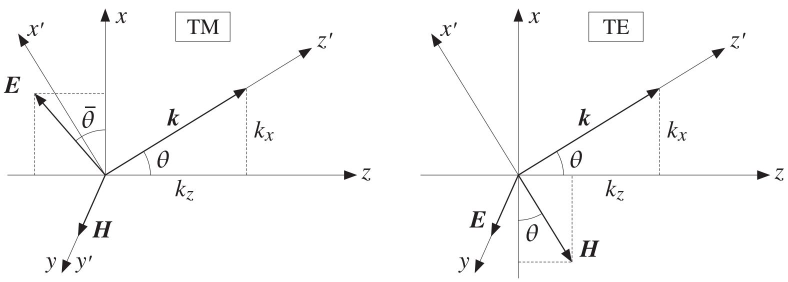

Fig. 3.6.1 Uniform plane waves in a birefringent medium.

For a biaxial medium, the three n_i are all different. For a uniaxial medium, we take the xy-axes to be *ordinary*, with $n_1 = n_2 = n_o$, and the z-axis to be *extraordinary*, with $n_3 = n_e$.† The wave vector $\mathbf{k}$ can be resolved along the z and x directions as follows:

$$\mathbf{k} = k\,\hat{\mathbf{k}} = k\,(\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta) = \hat{\mathbf{x}}k_x + \hat{\mathbf{z}}k_z \tag{3.6.6}$$

The ω-k relationship is determined from the solution of Maxwell's equations. By analogy with the isotropic case that has $k = nk_0 = n\omega/c_0$, we may define an *effective* refractive index N such that:

$$\boxed{k = Nk_0 = N\,\frac{\omega}{c_0}} \qquad \text{(effective refractive index)} \tag{3.6.7}$$

We will see in Eq. (3.6.22) by solving Maxwell's equations that N depends on the chosen polarization (according to Fig. 3.6.1) and on the wave vector direction θ:

$$N = \begin{cases} \dfrac{n_1 n_3}{\sqrt{n_1^2\sin^2\theta + n_3^2\cos^2\theta}}\,, & \text{(TM or p-polarization)} \\ n_2\,, & \text{(TE or s-polarization)} \end{cases} \tag{3.6.8}$$

For the TM case, we may rewrite the N-θ relationship in the form:

$$\boxed{\frac{1}{N^2} = \frac{\cos^2\theta}{n_1^2} + \frac{\sin^2\theta}{n_3^2}} \qquad \text{(effective TM index)} \tag{3.6.9}$$

Multiplying by k^2 and using $k_0 = k/N$, and $k_x = k\sin\theta$, $k_z = k\cos\theta$, we obtain the ω-k relationship for the TM case:

$$\frac{\omega^2}{c_0^2} = \frac{k_z^2}{n_1^2} + \frac{k_x^2}{n_3^2} \qquad \text{(TM or p-polarization)} \tag{3.6.10}$$

Similarly, we have for the TE case:

$$\frac{\omega^2}{c_0^2} = \frac{k^2}{n_2^2} \qquad \text{(TE or s-polarization)} \tag{3.6.11}$$

†In Sec. 3.2, the extraordinary axis was the x-axis.

Thus, the TE mode propagates as if the medium were isotropic with index $n = n_2$, whereas the TM mode propagates in a more complicated fashion. If the wave vector $\mathbf{k}$ is along the ordinary x-axis ($\theta = 90^o$), then $k = k_x = n_3\omega/c_0$ (this was the result of the previous section), and if $\mathbf{k}$ is along the extraordinary z-axis ($\theta = 0^o$), then we have $k = k_z = n_1\omega/c_0$.

For TM modes, the group velocity is not along $\mathbf{k}$. In general, the group velocity depends on the ω-k relationship and is computed as $\mathbf{v} = \partial\omega/\partial\mathbf{k}$. From Eq. (3.6.10), we find the x- and z-components:

$$\begin{aligned} v_x &= \frac{\partial\omega}{\partial k_x} = \frac{k_x c_0^2}{\omega n_3^2} = c_0\,\frac{N}{n_3^2}\,\sin\theta \\ v_z &= \frac{\partial\omega}{\partial k_z} = \frac{k_z c_0^2}{\omega n_1^2} = c_0\,\frac{N}{n_1^2}\,\cos\theta \end{aligned} \tag{3.6.12}$$

The velocity vector $\mathbf{v}$ is not parallel to $\mathbf{k}$. The angle $\bar\theta$ that $\mathbf{v}$ forms with the z-axis is given by $\tan\bar\theta = v_x/v_z$. It follows from (3.6.12) that:

$$\boxed{\tan\bar\theta = \frac{n_1^2}{n_3^2}\,\tan\theta} \qquad \text{(group velocity direction)} \tag{3.6.13}$$

Clearly, $\bar\theta \neq \theta$ if $n_1 \neq n_3$. The relative directions of $\mathbf{k}$ and $\mathbf{v}$ are shown in Fig. 3.6.2. The group velocity is also equal to the energy transport velocity defined in terms of the Poynting vector $\boldsymbol{\mathcal{P}}$ and energy density w as $\mathbf{v} = \boldsymbol{\mathcal{P}}/w$. Thus, $\mathbf{v}$ and $\boldsymbol{\mathcal{P}}$ have the same direction. Moreover, with the electric field being orthogonal to the Poynting vector, the angle $\bar\theta$ is also equal to the angle the $\mathbf{E}$-field forms with the x-axis.

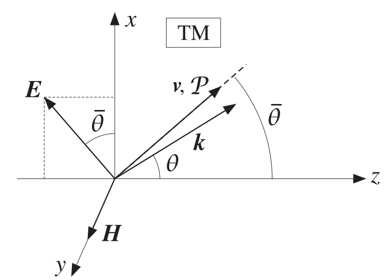

Fig. 3.6.2 Directions of group velocity, Poynting vector, wave vector, and electric field.

Next, we derive Eqs. (3.6.8) for N and solve for the field components in the TM and TE cases. We look for propagating solutions of Maxwell's equations of the type $\mathbf{E}(\mathbf{r}) = \mathbf{E}e^{-j\,\mathbf{k}\cdot\mathbf{r}}$ and $\mathbf{H}(\mathbf{r}) = \mathbf{H}e^{-j\,\mathbf{k}\cdot\mathbf{r}}$. Replacing the gradient operator by $\nabla \to -j\mathbf{k}$ and canceling some factors of j, Maxwell's equations take the form:

$$\begin{aligned}
\nabla\times\boldsymbol{E} &= -j\omega\mu_0\boldsymbol{H} & \qquad \boldsymbol{k}\times\boldsymbol{E} &= \omega\mu_0\boldsymbol{H}\\
\nabla\times\boldsymbol{H} &= j\omega\boldsymbol{D} & \Rightarrow \qquad \boldsymbol{k}\times\boldsymbol{H} &= -\omega\boldsymbol{D}\\
\nabla\cdot\boldsymbol{D} &= 0 & \boldsymbol{k}\cdot\boldsymbol{D} &= 0\\
\nabla\cdot\boldsymbol{H} &= 0 & \boldsymbol{k}\cdot\boldsymbol{H} &= 0
\end{aligned} \tag{3.6.14}$$

The last two equations are implied by the first two, as can be seen by dotting both sides of the first two with $\boldsymbol{k}$. Replacing $\boldsymbol{k} = k\,\hat{\mathbf{k}} = Nk_0\hat{\mathbf{k}}$, where N is still to be determined, we may solve Faraday's law for $\boldsymbol{H}$ in terms of $\boldsymbol{E}$:

$$N\frac{\omega}{c_0}\,\hat{\mathbf{k}}\times\boldsymbol{E} = \omega\mu_0\boldsymbol{H} \quad\Rightarrow\quad \boxed{\boldsymbol{H} = \frac{N}{\eta_0}\,\hat{\mathbf{k}}\times\boldsymbol{E}} \tag{3.6.15}$$

where we used $\eta_0 = c_0\mu_0$. Then, Ampère's law gives:

$$\boldsymbol{D} = -\frac{1}{\omega}\,\boldsymbol{k}\times\boldsymbol{H} = -\frac{1}{\omega}N\frac{\omega}{c_0}\,\hat{\mathbf{k}}\times\boldsymbol{H} = \frac{N^2}{\eta_0 c_0}\,\hat{\mathbf{k}}\times(\boldsymbol{E}\times\hat{\mathbf{k}}) \quad\Rightarrow\quad \boxed{\hat{\mathbf{k}}\times(\boldsymbol{E}\times\hat{\mathbf{k}}) = \frac{1}{\epsilon_0 N^2}\,\boldsymbol{D}}$$

where we used $c_0\eta_0 = 1/\epsilon_0$. The quantity $\hat{\mathbf{k}}\times(\boldsymbol{E}\times\hat{\mathbf{k}})$ is recognized as the component of $\boldsymbol{E}$ that is transverse to the propagation unit vector $\hat{\mathbf{k}}$. Using the BAC-CAB vector identity, we have $\hat{\mathbf{k}}\times(\boldsymbol{E}\times\hat{\mathbf{k}}) = \boldsymbol{E} - \hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\boldsymbol{E})$. Rearranging terms, we obtain:

$$\boxed{\boldsymbol{E} - \frac{1}{\epsilon_0 N^2}\,\boldsymbol{D} = \hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\boldsymbol{E})} \tag{3.6.16}$$

Because $\boldsymbol{D}$ is linear in $\boldsymbol{E}$, this is a homogeneous linear equation. Therefore, in order to have a nonzero solution, its determinant must be zero. This provides a condition from which N can be determined.

To obtain both the TE and TM solutions, we assume initially that $\boldsymbol{E}$ has all its three components and rewrite Eq. (3.6.16) component-wise. Using Eq. (3.6.5) and noting that $\hat{\mathbf{k}}\cdot\boldsymbol{E} = E_x\sin\theta + E_z\cos\theta$, we obtain the homogeneous linear system:

$$\begin{aligned}
\Big(1 - \frac{n_1^2}{N^2}\Big)E_x &= (E_x\sin\theta + E_z\cos\theta)\sin\theta\\
\Big(1 - \frac{n_2^2}{N^2}\Big)E_y &= 0\\
\Big(1 - \frac{n_3^2}{N^2}\Big)E_z &= (E_x\sin\theta + E_z\cos\theta)\cos\theta
\end{aligned} \tag{3.6.17}$$

The TE case has $E_y \neq 0$ and $E_x = E_z = 0$, whereas the TM case has $E_x \neq 0$, $E_z \neq 0$, and $E_y = 0$. Thus, the two cases decouple.

In the TE case, the second of Eqs. (3.6.17) immediately implies that $N = n_2$. Setting $\boldsymbol{E} = E_0\hat{\mathbf{y}}$ and using $\hat{\mathbf{k}}\times\hat{\mathbf{y}} = -\hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta$, we obtain the TE solution:

$$\boxed{\begin{aligned}
\boldsymbol{E}(\mathbf{r}) &= E_0\,\hat{\mathbf{y}}\,e^{-j\,\boldsymbol{k}\cdot\mathbf{r}}\\
\boldsymbol{H}(\mathbf{r}) &= \frac{n_2}{\eta_0}\,E_0\,(-\hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta)\,e^{-j\,\boldsymbol{k}\cdot\mathbf{r}}
\end{aligned}} \quad \text{(TE)} \tag{3.6.18}$$

where the TE propagation phase factor is:

$$\boxed{e^{-j\,\boldsymbol{k}\cdot\mathbf{r}} = e^{-jk_0 n_2(z\cos\theta + x\sin\theta)}} \quad \text{(TE propagation factor)} \tag{3.6.19}$$

The TM case requires a little more work. The linear system (3.6.17) becomes now:

$$\begin{aligned}
\Big(1 - \frac{n_1^2}{N^2}\Big)E_x &= (E_x\sin\theta + E_z\cos\theta)\sin\theta\\
\Big(1 - \frac{n_3^2}{N^2}\Big)E_z &= (E_x\sin\theta + E_z\cos\theta)\cos\theta
\end{aligned} \tag{3.6.20}$$

Using the identity $\sin^2\theta + \cos^2\theta = 1$, we may rewrite Eq. (3.6.20) in the matrix form:

$$\begin{bmatrix} \cos^2\theta - \dfrac{n_1^2}{N^2} & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin^2\theta - \dfrac{n_3^2}{N^2} \end{bmatrix} \begin{bmatrix} E_x \\ E_z \end{bmatrix} = 0 \tag{3.6.21}$$

Setting the determinant of the coefficient matrix to zero, we obtain the desired condition on N in order that a non-zero solution E_x, E_z exist:

$$\Big(\cos^2\theta - \frac{n_1^2}{N^2}\Big)\Big(\sin^2\theta - \frac{n_3^2}{N^2}\Big) - \sin^2\theta\cos^2\theta = 0 \tag{3.6.22}$$

This can be solved for N^2 to give Eq. (3.6.9). From it, we may also derive the following relationship, which will prove useful in applying Snell's law in birefringent media:

$$\boxed{N\cos\theta = \frac{n_1}{n_3}\sqrt{n_3^2 - N^2\sin^2\theta} = n_1\sqrt{1 - \frac{N^2\sin^2\theta}{n_3^2}}} \tag{3.6.23}$$

With the help of the relationships given in Problem 3.16, the solution of the homogeneous system (3.6.20) is found to be, up to a proportionality constant:

$$E_x = A\,\frac{n_3}{n_1}\cos\theta\,, \quad E_z = -A\,\frac{n_1}{n_3}\sin\theta \tag{3.6.24}$$

The constant A can be expressed in terms of the total magnitude of the field $E_0 = |\boldsymbol{E}| = \sqrt{|E_x|^2 + |E_z|^2}$. Using the relationship (3.7.11), we find (assuming $A > 0$):

$$A = E_0\,\frac{N}{\sqrt{n_1^2 + n_3^2 - N^2}} \tag{3.6.25}$$

The magnetic field $\boldsymbol{H}$ can also be expressed in terms of the constant A. We have:

$$
\begin{aligned}
\boldsymbol{H} &= \frac{N}{\eta_0}\,\hat{\mathbf{k}}\times\boldsymbol{E} = \frac{N}{\eta_0}\,(\hat{\mathbf{x}}\sin\theta+\hat{\mathbf{z}}\cos\theta)\times(\hat{\mathbf{x}}E_x+\hat{\mathbf{z}}E_z)\\
&= \frac{N}{\eta_0}\,\hat{\mathbf{y}}\,(E_x\cos\theta - E_z\sin\theta) = \frac{N}{\eta_0}\,\hat{\mathbf{y}}\,A\left(\frac{n_3}{n_1}\,\cos^2\theta+\frac{n_1}{n_3}\,\sin^2\theta\right)\\
&= \frac{N}{\eta_0}\,\hat{\mathbf{y}}\,A\,\frac{n_1n_3}{N^2} = \frac{A}{\eta_0}\,\hat{\mathbf{y}}\,\frac{n_1n_3}{N}
\end{aligned}
\tag{3.6.26}
$$

where we used Eq. (3.7.10). In summary, the complete TM solution is:

$$
\boxed{
\begin{aligned}
\boldsymbol{E}(\mathbf{r}) &= E_0\,\frac{N}{\sqrt{n_1^2+n_3^2-N^2}}\left(\hat{\mathbf{x}}\,\frac{n_3}{n_1}\,\cos\theta-\hat{\mathbf{z}}\,\frac{n_1}{n_3}\,\sin\theta\right)e^{-j\,\boldsymbol{k}\cdot\mathbf{r}}\\
\boldsymbol{H}(\mathbf{r}) &= \frac{E_0}{\eta_0}\,\frac{n_1n_3}{\sqrt{n_1^2+n_3^2-N^2}}\,\hat{\mathbf{y}}\,e^{-j\,\boldsymbol{k}\cdot\mathbf{r}}
\end{aligned}
}
\quad \text{(TM)}
\tag{3.6.27}
$$

where the TM propagation phase factor is:

$$
\boxed{e^{-j\,\boldsymbol{k}\cdot\mathbf{r}} = e^{-jk_0N(z\cos\theta+x\sin\theta)}} \quad \text{(TM propagation factor)}
\tag{3.6.28}
$$

The solution has been put in a form that exhibits the proper limits at $\theta = 0^{\text{o}}$ and 90^{o}. It agrees with Eq. (3.6.3) in the isotropic case. The angle that $\boldsymbol{E}$ forms with the x-axis in Fig. 3.6.2 is given by $\tan\bar{\theta} = -E_z/E_x$ and agrees with Eq. (3.6.13).

Next, we derive expressions for the Poynting vector and energy densities. It turns out—as is common in propagation and waveguide problems—that the magnetic energy density is equal to the electric one. Using Eq. (3.6.27), we find:

$$
\boldsymbol{\mathcal{P}} = \frac{1}{2}\,\text{Re}(\boldsymbol{E}\times\boldsymbol{H}^*) = \frac{E_0^2}{2\eta_0}\,\frac{n_1n_3N}{n_1^2+n_3^2-N^2}\left(\hat{\mathbf{x}}\,\frac{n_1}{n_3}\,\sin\theta+\hat{\mathbf{z}}\,\frac{n_3}{n_1}\,\cos\theta\right)
\tag{3.6.29}
$$

and for the electric, magnetic, and total energy densities:

$$
\begin{aligned}
w_e &= \frac{1}{2}\,\text{Re}(\boldsymbol{D}\cdot\boldsymbol{E}^*) = \frac{1}{4}\epsilon_0\big(n_1^2|E_x|^2+n_3^2|E_z|^2\big) = \frac{1}{4}\epsilon_0E_0^2\,\frac{n_1^2n_3^2}{n_1^2+n_3^2-N^2}\\
w_m &= \frac{1}{2}\,\text{Re}(\boldsymbol{B}\cdot\boldsymbol{H}^*) = \frac{1}{4}\mu_0|H_y|^2 = \frac{1}{4}\epsilon_0E_0^2\,\frac{n_1^2n_3^2}{n_1^2+n_3^2-N^2} = w_e\\
w &= w_e+w_m = 2w_e = \frac{1}{2}\epsilon_0E_0^2\,\frac{n_1^2n_3^2}{n_1^2+n_3^2-N^2}
\end{aligned}
\tag{3.6.30}
$$

The vector $\boldsymbol{\mathcal{P}}$ is orthogonal to $\boldsymbol{E}$ and its direction is $\bar{\theta}$ given by Eq. (3.6.13), as can be verified by taking the ratio $\tan\bar{\theta} = \mathcal{P}_x/\mathcal{P}_z$. The energy transport velocity is the ratio of the energy flux to the energy density—it agrees with the group velocity (3.6.12):

$$
\boldsymbol{v} = \frac{\boldsymbol{\mathcal{P}}}{w} = c_0\left(\hat{\mathbf{x}}\,\frac{N}{n_3^2}\,\sin\theta+\hat{\mathbf{z}}\,\frac{N}{n_1^2}\,\cos\theta\right)
\tag{3.6.31}
$$

To summarize, the TE and TM uniform plane wave solutions are given by Eqs. (3.6.18) and (3.6.27). We will use these results in Sects. 7.7 and 7.9 to discuss reflection and refraction in birefringent media and multilayer birefringent dielectric structures. Further discussion of propagation in birefringent media can be found in [205] and [264–284].

3.7 Problems

3.1 For the circular-polarization basis of Eq. (3.1.1), show

$$
\boldsymbol{E} = \hat{\mathbf{e}}_+E_+ + \hat{\mathbf{e}}_-E_- \quad\Rightarrow\quad \hat{\mathbf{z}}\times\boldsymbol{E} = j\,\hat{\mathbf{e}}_+E_+ - j\,\hat{\mathbf{e}}_-E_- \quad\Rightarrow\quad \hat{\mathbf{z}}\times\boldsymbol{E}_\pm = \pm j\boldsymbol{E}_\pm
$$

3.2 Show the component-wise Maxwell equations, Eqs. (3.1.4) and (3.1.5), with respect to the linear and circular polarization bases.

3.3 Suppose that the two unit vectors $\{\hat{\mathbf{x}},\hat{\mathbf{y}}\}$ are rotated about the z-axis by an angle ϕ resulting in $\hat{\mathbf{x}}' = \hat{\mathbf{x}}\cos\phi+\hat{\mathbf{y}}\sin\phi$ and $\hat{\mathbf{y}}' = \hat{\mathbf{y}}\cos\phi-\hat{\mathbf{x}}\sin\phi$. Show that the corresponding circular basis vectors $\hat{\mathbf{e}}_\pm = \hat{\mathbf{x}}\mp j\,\hat{\mathbf{y}}$ and $\hat{\mathbf{e}}'_\pm = \hat{\mathbf{x}}'\mp j\,\hat{\mathbf{y}}'$ change by the phase factors: $\hat{\mathbf{e}}'_\pm = e^{\pm j\phi}\hat{\mathbf{e}}_\pm$.

3.4 Consider a linearly birefringent 90^{o} quarter-wave retarder. Show that the following input polarizations change into the indicated output ones:

$$
\begin{aligned}
\hat{\mathbf{x}}\pm\hat{\mathbf{y}} \quad&\to\quad \hat{\mathbf{x}}\pm j\,\hat{\mathbf{y}}\\
\hat{\mathbf{x}}\pm j\,\hat{\mathbf{y}} \quad&\to\quad \hat{\mathbf{x}}\pm\hat{\mathbf{y}}
\end{aligned}
$$

What are the output polarizations if the same input polarizations go through a 180^{o} half-wave retarder?

3.5 A polarizer lets through linearly polarized light in the direction of the unit vector $\hat{\mathbf{e}}_p = \hat{\mathbf{x}}\cos\theta_p+\hat{\mathbf{y}}\sin\theta_p$, as shown in Fig. 3.7.1. The output of the polarizer propagates in the z-direction through a linearly birefringent retarder of length l, with birefringent refractive indices n_1, n_2, and retardance $\phi = (n_1-n_2)k_0l$.

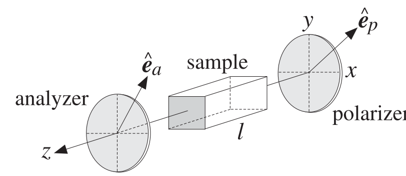

Fig. 3.7.1 Polarizer-analyzer measurement of birefringence.

The output $\boldsymbol{E}(l)$ of the birefringent sample goes through an analyzing linear polarizer that lets through polarizations along the unit vector $\hat{\mathbf{e}}_a = \hat{\mathbf{x}}\cos\theta_a+\hat{\mathbf{y}}\sin\theta_a$. Show that the light intensity at the output of the analyzer is given by:

$$
I_a = \big|\hat{\mathbf{e}}_a\cdot\boldsymbol{E}(l)\big|^2 = \big|\cos\theta_a\cos\theta_p + e^{j\phi}\sin\theta_a\sin\theta_p\big|^2
$$

For a circularly birefringent sample that introduces a natural or Faraday rotation of $\phi = (k_+-k_-)l/2$, show that the output light intensity will be:

$$
I_a = \big|\hat{\mathbf{e}}_a\cdot\boldsymbol{E}(l)\big|^2 = \cos^2(\theta_p-\theta_a-\phi)
$$

For both the linear and circular cases, what are some convenient choices for θ_a and θ_p?

3.6 A linearly polarized wave with polarization direction at an angle θ with the x-axis goes through a circularly birefringent retarder that introduces an optical rotation by the angle $\phi = (k_+-k_-)l/2$. Show that the input and output polarization directions will be:

$$
\hat{\mathbf{x}}\cos\theta+\hat{\mathbf{y}}\sin\theta \quad\to\quad \hat{\mathbf{x}}\cos(\theta-\phi)+\hat{\mathbf{y}}\sin(\theta-\phi)
$$

3.7 Show that an arbitrary polarization vector can be expressed as follows with respect to a linear basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ and its rotated version $\{\hat{\mathbf{x}}', \hat{\mathbf{y}}'\}$:

$$\boldsymbol{E} = A\,\hat{\mathbf{x}} + B\,\hat{\mathbf{y}} = A'\,\hat{\mathbf{x}}' + B'\,\hat{\mathbf{y}}'$$

where the new coefficients and the new basis vectors are related to the old ones by a rotation by an angle ϕ:

$$\begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}, \quad \begin{bmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{bmatrix}$$

3.8 Show that the source-free Maxwell's equations (3.1.2) for a chiral medium characterized by (3.3.1), may be cast in the matrix form, where $k = \omega\sqrt{\mu\epsilon}$, $\eta = \sqrt{\mu/\epsilon}$, and $a = \chi/\sqrt{\mu\epsilon}$:

$$\nabla\times\begin{bmatrix} \boldsymbol{E} \\ \eta\boldsymbol{H} \end{bmatrix} = \begin{bmatrix} ka & -jk \\ jk & ka \end{bmatrix} \begin{bmatrix} \boldsymbol{E} \\ \eta\boldsymbol{H} \end{bmatrix}$$

Show that these may be decoupled by forming the "right" and "left" polarized fields:

$$\nabla\times\begin{bmatrix} \boldsymbol{E}_R \\ \boldsymbol{E}_L \end{bmatrix} = \begin{bmatrix} k_+ & 0 \\ 0 & -k_- \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_R \\ \boldsymbol{E}_L \end{bmatrix}, \quad \text{where} \quad \boldsymbol{E}_R = \frac{1}{2}(\boldsymbol{E} - j\eta\boldsymbol{H}), \quad \boldsymbol{E}_L = \frac{1}{2}(\boldsymbol{E} + j\eta\boldsymbol{H})$$

where $k_\pm = k(1 \pm a)$. Using these results, show that the possible plane-wave solutions propagating in the direction of a unit-vector $\hat{\mathbf{k}}$ are given by:

$$\boldsymbol{E}(r) = E_0(\hat{\mathbf{p}} - j\,\hat{\mathbf{s}})\,e^{-j\,\mathbf{k}_+\cdot\mathbf{r}} \quad \text{and} \quad \boldsymbol{E}(r) = E_0(\hat{\mathbf{p}} + j\,\hat{\mathbf{s}})\,e^{-j\,\mathbf{k}_-\cdot\mathbf{r}}$$

where $\boldsymbol{k}_\pm = k_\pm\,\hat{\mathbf{k}}$ and $\{\hat{\mathbf{p}}, \hat{\mathbf{s}}, \hat{\mathbf{k}}\}$ form a right-handed system of unit vectors, such as $\{\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'\}$ of Fig. 2.9.1. Determine expressions for the corresponding magnetic fields. What freedom do we have in selecting $\{\hat{\mathbf{p}}, \hat{\mathbf{s}}\}$ for a given direction $\hat{\mathbf{k}}$?

3.9 Using Maxwell's equations (3.1.2), show the following Poynting-vector relationships for an arbitrary source-free medium:

$$\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}^*) = j\omega(\boldsymbol{D}^*\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{H}^*)$$

$$\nabla\cdot\mathrm{Re}(\boldsymbol{E}\times\boldsymbol{H}^*) = -\omega\,\mathrm{Im}(\boldsymbol{D}^*\cdot\boldsymbol{E} + \boldsymbol{B}^*\cdot\boldsymbol{H})$$

Explain why a lossless medium must satisfy the condition $\nabla\cdot\mathrm{Re}(\boldsymbol{E}\times\boldsymbol{H}^*) = 0$. Show that this condition requires that the energy function $w = (\boldsymbol{D}^*\cdot\boldsymbol{E} + \boldsymbol{B}^*\cdot\boldsymbol{H})/2$ be real-valued.

For a lossless chiral medium characterized by (3.3.1), show that the parameters ϵ, μ, χ are required to be real. Moreover, show that the positivity of the energy function $w > 0$ requires that $|\chi| < \sqrt{\mu\epsilon}$, as well as $\epsilon > 0$ and $\mu > 0$.

3.10 In a chiral medium, at $z = 0$ we lauch the fields $E_{R+}(0)$ and $E_{L-}(0)$, which propagate by a distance l, get reflected, and come back to the starting point. Assume that at the point of reversal the fields remain unchanged, that is, $E_{R+}(l) = E_{L+}(l)$ and $E_{L-}(l) = E_{R-}(l)$. Using the propagation results (3.3.5) and (3.3.9), show that fields returned back at $z = 0$ will be:

$$E_{L+}(0) = E_{L+}(l)\,e^{-jk_-l} = E_{R+}(l)\,e^{-jk_-l} = E_{R+}(0)\,e^{-j(k_++k_-)l}$$
$$E_{R-}(0) = E_{R-}(l)\,e^{-jk_+l} = E_{L-}(l)\,e^{-jk_+l} = E_{L-}(0)\,e^{-j(k_++k_-)l}$$

Show that the overall natural rotation angle will be zero. For a gyrotropic medium, show that the corresponding rountrip fields will be:

$$E_{L+}(0) = E_{L+}(l)\,e^{-jk_-l} = E_{R+}(l)\,e^{-jk_-l} = E_{R+}(0)\,e^{-2jk_+l}$$
$$E_{R-}(0) = E_{R-}(l)\,e^{-jk_+l} = E_{L-}(l)\,e^{-jk_+l} = E_{L-}(0)\,e^{-2jk_-l}$$

Show that the total Faraday rotation angle will be $2\phi = (k_+ - k_-)l$.

3.11 Show that the x, y components of the gyroelectric and gyromagnetic constitutive relationships (3.4.1) and (3.4.2) may be written in the compact forms:

$$\boldsymbol{D}_T = \epsilon_1\boldsymbol{E}_T - j\epsilon_2\,\hat{\mathbf{z}}\times\boldsymbol{E}_T \qquad \text{(gyroelectric)}$$
$$\boldsymbol{B}_T = \mu_1\boldsymbol{H}_T - j\mu_2\,\hat{\mathbf{z}}\times\boldsymbol{H}_T \qquad \text{(gyromagnetic)}$$

where the subscript T indicates the transverse (with respect to z) part of a vector, for example, $\boldsymbol{D}_T = \hat{\mathbf{x}}\,D_x + \hat{\mathbf{y}}\,D_y$.

3.12 Conductors and plasmas exhibit gyroelectric behavior when they are in the presence of an external magnetic field. The equation of motion of conduction electrons in a constant magnetic field is $m\dot{\boldsymbol{v}} = e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) - m\alpha\boldsymbol{v}$, with the collisional damping term included. The magnetic field is in the z-direction, $\boldsymbol{B} = \hat{\mathbf{z}}\,B_0$.

Assuming $e^{j\omega t}$ time dependence and decomposing all vectors in the circular basis (3.1.1), for example, $\boldsymbol{v} = \hat{\mathbf{e}}_+ v_+ + \hat{\mathbf{e}}_- v_- + \hat{\mathbf{z}}\,v_z$, show that the solution of the equation of motion is:

$$v_\pm = \frac{\dfrac{e}{m}E_\pm}{\alpha + j(\omega \pm \omega_B)}, \quad v_z = \frac{\dfrac{e}{m}E_z}{\alpha + j\omega}$$

where $\omega_B = eB_0/m$ is the cyclotron frequency. Then, show that the D–E constitutive relationship takes the form of Eq. (3.4.1) with:

$$\epsilon_\pm = \epsilon_1 \pm \epsilon_2 = \epsilon_0\left[1 - \frac{j\omega_p^2}{\omega[\alpha + j(\omega \pm \omega_B)]}\right], \quad \epsilon_3 = \epsilon_0\left[1 - \frac{j\omega_p^2}{\omega(\alpha + j\omega)}\right]$$

where $\omega_p^2 = Ne^2/m\epsilon_0$ is the plasma frequency and N, the number of conduction electrons per unit volume. (See Problem 1.9 for some helpful hints.)

3.13 If the magnetic field $\boldsymbol{H}_{\text{tot}} = \hat{\mathbf{z}}\,H_0 + \boldsymbol{H}e^{j\omega t}$ is applied to a magnetizable sample, the induced magnetic moment per unit volume (the magnetization) will have the form $\boldsymbol{M}_{\text{tot}} = \hat{\mathbf{z}}\,M_0 + \boldsymbol{M}e^{j\omega t}$, where $\hat{\mathbf{z}}\,M_0$ is the saturation magnetization due to $\hat{\mathbf{z}}\,H_0$ acting alone. The phenomenological equations governing $\boldsymbol{M}_{\text{tot}}$, including a so-called Landau-Lifshitz damping term, are given by [315]:

$$\frac{d\boldsymbol{M}_{\text{tot}}}{dt} = \gamma(\boldsymbol{M}_{\text{tot}}\times\boldsymbol{H}_{\text{tot}}) - \frac{\alpha}{M_0H_0}\boldsymbol{M}_{\text{tot}}\times(\boldsymbol{M}_{\text{tot}}\times\boldsymbol{H}_{\text{tot}})$$

where γ is the gyromagnetic ratio and $\tau = 1/\alpha$, a relaxation time constant. Assuming that $|\boldsymbol{H}| \ll H_0$ and $|\boldsymbol{M}| \ll M_0$, show that the linearized version of this equation obtained by keeping only first order terms in $\boldsymbol{H}$ and $\boldsymbol{M}$ is:

$$j\omega\boldsymbol{M} = \omega_M(\hat{\mathbf{z}}\times\boldsymbol{H}) - \omega_H(\hat{\mathbf{z}}\times\boldsymbol{M}) - \alpha\,\hat{\mathbf{z}}\times\left[(\boldsymbol{M} - \chi_0\boldsymbol{H})\times\hat{\mathbf{z}}\right]$$

where $\omega_M = \gamma M_0$, $\omega_H = \gamma H_0$, and $\chi_0 = M_0/H_0$. Working in the circular basis (3.1.1), show that the solution of this equation is:

$$M_\pm = \chi_0\,\frac{\alpha \pm j\omega_H}{\alpha + j(\omega \pm \omega_H)}\,H_\pm \equiv \chi_\pm H_\pm \quad\text{and}\quad M_z = 0$$

Writing $\boldsymbol{B} = \mu_0(\boldsymbol{H}+\boldsymbol{M})$, show that the permeability matrix has the gyromagnetic form of Eq. (3.4.2) with $\mu_1 \pm \mu_2 = \mu_\pm = \mu_0(1+\chi_\pm)$ and $\mu_3 = \mu_0$. Show that the real and imaginary parts of μ_1 are given by [315]:

$$\mathrm{Re}(\mu_1) = \mu_0 + \frac{\mu_0\chi_0}{2}\left[\frac{\alpha^2 + \omega_H(\omega+\omega_H)}{\alpha^2 + (\omega+\omega_H)^2} + \frac{\alpha^2 - \omega_H(\omega-\omega_H)}{\alpha^2 + (\omega-\omega_H)^2}\right]$$

$$\mathrm{Im}(\mu_1) = -\frac{\mu_0\chi_0}{2}\left[\frac{\alpha\omega}{\alpha^2 + (\omega+\omega_H)^2} + \frac{\alpha\omega}{\alpha^2 + (\omega-\omega_H)^2}\right]$$

Derive similar expressions for $\mathrm{Re}(\mu_2)$ and $\mathrm{Im}(\mu_2)$.

3.14 A uniform plane wave, $\boldsymbol{E}e^{-j\boldsymbol{k}\cdot\mathbf{r}}$ and $\boldsymbol{H}e^{-j\boldsymbol{k}\cdot\mathbf{r}}$, is propagating in the direction of the unit vector $\hat{\mathbf{k}} = \hat{\mathbf{z}}' = \hat{\mathbf{z}}\cos\theta + \hat{\mathbf{z}}\sin\theta$ shown in Fig. 2.9.1 in a gyroelectric medium with constitutive relationships (3.4.1).

Show that Eqs. (3.6.14)–(3.6.16) remain valid provided we define the effective refractive index N through the wavevector $\boldsymbol{k} = k\hat{\mathbf{k}}$, where $k = Nk_0$, $k_0 = \omega\sqrt{\mu\epsilon_0}$.

Working in the circular-polarization basis (3.1.1), that is, $\boldsymbol{E} = \hat{\mathbf{e}}_+E_+ + \hat{\mathbf{e}}_-E_- + \hat{\mathbf{z}}E_z$, where $E_\pm = (E_x \pm jE_y)/2$, show that Eq. (3.6.16) leads to the homogeneous system:

$$\begin{bmatrix} 1 - \frac{1}{2}\sin^2\theta - \frac{\epsilon_+}{\epsilon_0 N^2} & -\frac{1}{2}\sin^2\theta & -\frac{1}{2}\sin\theta\cos\theta \\ -\frac{1}{2}\sin^2\theta & 1 - \frac{1}{2}\sin^2\theta - \frac{\epsilon_-}{\epsilon_0 N^2} & -\frac{1}{2}\sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\sin\theta\cos\theta & \sin^2\theta - \frac{\epsilon_3}{\epsilon_0 N^2}\end{bmatrix}\begin{bmatrix}E_+\\E_-\\E_z\end{bmatrix} = 0 \tag{3.7.1}$$

where $\epsilon_\pm = \epsilon_1 \pm \epsilon_2$. Alternatively, show that in the linear-polarization basis:

$$\begin{bmatrix} \epsilon_1 - \epsilon_0 N^2\cos^2\theta & j\epsilon_2 & \epsilon_0 N^2\sin\theta\cos\theta \\ -j\epsilon_2 & \epsilon_1 - \epsilon_0 N^2 & 0 \\ \epsilon_0 N^2 \sin\theta\cos\theta & 0 & \epsilon_3 - \epsilon_0 N^2\sin^2\theta\end{bmatrix}\begin{bmatrix}E_x\\E_y\\E_z\end{bmatrix} = 0 \tag{3.7.2}$$

For either basis, setting the determinant of the coefficient matrix to zero, show that a non-zero $\boldsymbol{E}$ solution exists provided that N^2 is one of the two solutions of:

$$\tan^2\theta = -\frac{\epsilon_3}{\epsilon_1}\,\frac{(\epsilon_0N^2 - \epsilon_+)(\epsilon_0N^2-\epsilon_-)}{(\epsilon_0N^2-\epsilon_3)(\epsilon_0N^2-\epsilon_e)}\,,\quad\text{where}\quad \epsilon_e = \frac{2\epsilon_+\epsilon_-}{\epsilon_+ + \epsilon_-} = \frac{\epsilon_1^2 - \epsilon_2^2}{\epsilon_1} \tag{3.7.3}$$

Show that the two solutions for N^2 are:

$$N^2 = \frac{(\epsilon_1^2 - \epsilon_2^2 - \epsilon_1\epsilon_3)\sin^2\theta + 2\epsilon_1\epsilon_3 \pm \sqrt{(\epsilon_1^2 - \epsilon_2^2 - \epsilon_1\epsilon_3)^2\sin^4\theta + 4\epsilon_2^2\epsilon_3^2\cos^2\theta}}{2\epsilon_0(\epsilon_1\sin^2\theta + \epsilon_3\cos^2\theta)} \tag{3.7.4}$$

For the special case $\hat{\mathbf{k}} = \hat{\mathbf{z}}$ ($\theta = 0^\mathrm{o}$), show that the two possible solutions of Eq. (3.7.1) are:

$$\begin{aligned}&\epsilon_0N^2 = \epsilon_+\,, && k = k_+ = \omega\sqrt{\mu\epsilon_+}\,, && E_+ \neq 0, && E_- = 0, && E_z = 0\\ &\epsilon_0N^2 = \epsilon_-\,, && k = k_- = \omega\sqrt{\mu\epsilon_-}\,, && E_+ = 0, && E_- \neq 0, && E_z = 0\end{aligned}$$

For the case $\hat{\mathbf{k}} = \hat{\mathbf{x}}$ ($\theta = 90^\mathrm{o}$), show that:

$$\begin{aligned}&\epsilon_0N^2 = \epsilon_3\,, && k = k_3 = \omega\sqrt{\mu\epsilon_3}\,, && E_+ = 0, && E_- = 0, && E_z \neq 0\\ &\epsilon_0N^2 = \epsilon_e\,, && k = k_e = \omega\sqrt{\mu\epsilon_e}\,, && E_+ \neq 0, && E_- = -\frac{\epsilon_+}{\epsilon_-}E_+\,, && E_z = 0\end{aligned}$$

For each of the above four special solutions, derive the corresponding magnetic fields $\boldsymbol{H}$. Justify the four values of N^2 on the basis of Eq. (3.7.3). Discuss the polarization properties of the four cases. For the "extraordinary" wave $k = k_e$, show that $D_x = 0$ and $E_x/E_y = -j\epsilon_2/\epsilon_1$.

Eq. (3.7.4) and the results of Problem 3.14 lead to the so-called Appleton-Hartree equations for describing plasma waves in a magnetic field [302–306].

3.15 A uniform plane wave, $\boldsymbol{E}e^{-j\boldsymbol{k}\cdot\mathbf{r}}$ and $\boldsymbol{H}e^{-j\boldsymbol{k}\cdot\mathbf{r}}$, is propagating in the direction of the unit vector $\hat{\mathbf{k}} = \hat{\mathbf{z}}' = \hat{\mathbf{z}}\cos\theta + \hat{\mathbf{z}}\sin\theta$ shown in Fig. 2.9.1 in a gyromagnetic medium with constitutive relationships (3.4.2). Using Maxwell's equations, show that:

$$\begin{aligned}&\boldsymbol{k}\times\boldsymbol{E} = \omega\boldsymbol{B}, && \boldsymbol{k}\cdot\boldsymbol{B} = 0\\ &\boldsymbol{k}\times\boldsymbol{H} = -\omega\epsilon\boldsymbol{E}, && \boldsymbol{k}\cdot\boldsymbol{E} = 0\end{aligned}\quad\Rightarrow\quad \boldsymbol{H} - \frac{1}{\mu_0N^2}\boldsymbol{B} = \hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\boldsymbol{H}) \tag{3.7.5}$$

where the effective refractive index N is defined through the wavevector $\boldsymbol{k} = k\hat{\mathbf{k}}$, where $k = Nk_0$, $k_0 = \omega\sqrt{\mu_0\epsilon}$. Working in the circular polarization basis $\boldsymbol{H} = \hat{\mathbf{e}}_+H_+ + \hat{\mathbf{e}}_-H_- + \hat{\mathbf{z}}H_z$, where $H_\pm = (H_x \pm jH_y)/2$, show that Eq. (3.7.5) leads to the homogeneous system:

$$\begin{bmatrix} 1 - \frac{1}{2}\sin^2\theta - \frac{\mu_+}{\mu_0 N^2} & -\frac{1}{2}\sin^2\theta & -\frac{1}{2}\sin\theta\cos\theta \\ -\frac{1}{2}\sin^2\theta & 1 - \frac{1}{2}\sin^2\theta - \frac{\mu_-}{\mu_0 N^2} & -\frac{1}{2}\sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\sin\theta\cos\theta & \sin^2\theta - \frac{\mu_3}{\mu_0 N^2}\end{bmatrix}\begin{bmatrix}H_+\\H_-\\H_z\end{bmatrix} = 0 \tag{3.7.6}$$

where $\mu_\pm = \mu_1 \pm \mu_2$. Alternatively, show that in the linear-polarization basis:

$$\begin{bmatrix} \mu_1 - \mu_0 N^2\cos^2\theta & j\mu_2 & \mu_0 N^2\sin\theta\cos\theta \\ -j\mu_2 & \mu_1 - \mu_0 N^2 & 0 \\ \mu_0 N^2 \sin\theta\cos\theta & 0 & \mu_3 - \mu_0 N^2\sin^2\theta\end{bmatrix}\begin{bmatrix}H_x\\H_y\\H_z\end{bmatrix} = 0 \tag{3.7.7}$$

For either basis, setting the determinant of the coefficient matrix to zero, show that a non-zero $\boldsymbol{E}$ solution exists provided that N^2 is one of the two solutions of:

$$\tan^2\theta = -\frac{\mu_3}{\mu_1}\,\frac{(\mu_0N^2 - \mu_+)(\mu_0N^2-\mu_-)}{(\mu_0N^2-\mu_3)(\mu_0N^2-\mu_e)}\,,\quad\text{where}\quad \mu_e = \frac{2\mu_+\mu_-}{\mu_+ + \mu_-} = \frac{\mu_1^2 - \mu_2^2}{\mu_1} \tag{3.7.8}$$

Show that the two solutions for N^2 are:

$$N^2 = \frac{(\mu_1^2 - \mu_2^2 - \mu_1\mu_3)\sin^2\theta + 2\mu_1\mu_3 \pm \sqrt{(\mu_1^2 - \mu_2^2 - \mu_1\mu_3)^2\sin^4\theta + 4\mu_2^2\mu_3^2\cos^2\theta}}{2\mu_0(\mu_1\sin^2\theta + \mu_3\cos^2\theta)}$$

For the special case $\theta = 0^\mathrm{o}$, show that the two possible solutions of Eq. (3.7.6) are:

$$\begin{aligned}&\mu_0N^2 = \mu_+\,, && k = k_+ = \omega\sqrt{\epsilon\mu_+}\,, && H_+ \neq 0, && H_- = 0, && H_z = 0\\ &\mu_0N^2 = \mu_-\,, && k = k_+ = \omega\sqrt{\epsilon\mu_-}\,, && H_+ = 0, && H_- \neq 0, && H_z = 0\end{aligned}$$

For the special case $\theta = 90^\mathrm{o}$, show that:

$$\begin{aligned}&\mu_0N^2 = \mu_3\,, && k = k_3 = \omega\sqrt{\epsilon\mu_3}\,, && H_+ = 0, && H_- = 0, && H_z \neq 0\\ &\mu_0N^2 = \mu_e\,, && k = k_e = \omega\sqrt{\epsilon\mu_e}\,, && H_+ \neq 0, && H_- = -\frac{\mu_+}{\mu_-}H_+\,, && H_z = 0\end{aligned}$$

For each of the above four special solutions, derive the corresponding electric fields $\mathbf{E}$. Justify the four values of N^2 on the basis of Eq. (3.7.8). Discuss the polarization properties of the four cases. This problem is the dual of Problem 3.14.

3.16 Using Eq. (3.6.9) for the effective TM refractive index in a birefringent medium, show the following additional relationships:

$$\frac{\sin^2\theta}{1-\dfrac{n_1^2}{N^2}}+\frac{\cos^2\theta}{1-\dfrac{n_3^2}{N^2}}=1 \tag{3.7.9}$$

$$\frac{n_3}{n_1}\cos^2\theta+\frac{n_1}{n_3}\sin^2\theta=\frac{n_1n_3}{N^2} \tag{3.7.10}$$

$$\frac{n_1^2}{n_3^2}\sin^2\theta+\frac{n_3^2}{n_1^2}\cos^2\theta=\frac{n_1^2+n_3^2-N^2}{N^2} \tag{3.7.11}$$

$$\sin^2\theta=\frac{1-\dfrac{n_1^2}{N^2}}{1-\dfrac{n_1^2}{n_3^2}}\,,\qquad \cos^2\theta=\frac{1-\dfrac{n_3^2}{N^2}}{1-\dfrac{n_3^2}{n_1^2}} \tag{3.7.12}$$

$$\cos^2\theta-\frac{n_1^2}{N^2}=-\frac{n_1^2}{n_3^2}\sin^2\theta\,,\qquad \sin^2\theta-\frac{n_3^2}{N^2}=-\frac{n_3^2}{n_1^2}\cos^2\theta \tag{3.7.13}$$

Using these relationships, show that the homogeneous linear system (3.6.20) can be simplified into the form:

$$E_x\frac{n_1}{n_3}\sin\theta=-E_z\frac{n_3}{n_1}\cos\theta\,,\quad E_z\frac{n_3}{n_1}\cos\theta=-E_x\frac{n_1}{n_3}\sin\theta$$